

THE
DOCTRINE
AND
APPLICATION
OF
FLUXIONS.

CONTAINING

(Besides what is common on the Subject)

A Number of NEW IMPROVEMENTS
in the THEORY.

AND

The SOLUTION of a Variety of New, and very
Interesting, Problems in different Branches of
the MATHEMATICKS.

PART II.

By THOMAS SIMPSON, F.R.S.

LONDON:

Printed for J. Nourse, opposite *Catherine-street*
in the Strand. MDCCL.



ERRATA.

Page 286. line 9. for *Dedree*, r. *Degree*; p. 287. l. last, for $\dot{K}$, r. $\dot{H}$; p. 288. l. last, for y , r. y^{1-m} ; p. 289. l. 20. for *into*, r. *by*; p. 291. l. 7. before *applicable*, r. *chiefly*; p. 293. l. 7. for *Art.* 14. r. *Art.* 252; p. 298. l. 8. for $2A^2 dx$, r. $2a^2 Ax$; p. 300. l. last, for $-\frac{x^9}{144a^8}$, r. $+\frac{x^9}{144a^8}$; p. 304. l. last but one, for n , r. p ; p. 237. l. 1. for d^{2v} , r. $d^{2v}A$; p. 350. l. 2. for k , r. $2k$; l. last, before EG, r. EF; p. 384. l. 10. for x^{n+m-1} , r. $x^{n+m-1}\dot{x}$; and for x^{n-m-1} , r. $x^{n-m-1}\dot{x}$; p. 436. l. 23. for $\dot{z}^2$, r. $\ddot{z}$; p. 465. l. 9. and 14. for $2t$ (in the Divisor) r. $2t^5$; p. 470. l. 1. for *Problem*, r. *Proposition*; p. 481. l. 7. dele $\times A$; p. 484. l. 5. for $\dot{x}$, r. $\dot{y}$; l. 12. for u , r. $\dot{u}$; l. 13. for uu , r. $\dot{u}\dot{u}$; l. 15. for $r-1$, read $r-m$; l. 20. for $\dot{y}\dot{y}$, r. $\ddot{y}$; p. 504. l. 18. for BT, r. BF; p. 513. l. 3. for a^2 (in the Divisor) r. x ; l. 6. for $\dot{x}$, r. $\mp \dot{x}$; also in l. 6. and 7. for $-$ read $+$ and for ∞ r. $-$; p. 514. l. 7. and 8. for $+$, r. $-$; p. 521. l. 28. 29. and 30. for AEB, r. ABE; p. 544. l. 10. for $OP^2 \&c.$ r. $AD(1) : AB(\sqrt{1+bb}) ::$
After Line 2. p. 354. add—when n is odd; and the First and Last when n is even.

Books lately Published, by the same Author:

I. **E**SSAYS on several curious and useful Subjects in Speculative and Mix'd Mathematicks; in which are explain'd the most difficult Problems of the first and second Books of Sir *Iaac Newton's Principia*: Being an useful Introduction to Learners for understanding that illustrious Author. In 4to. 1740. Price 6s. sewed.

II. The Doctrine of Annuities and Reversions, deduced from general and evident Principles; with useful Tables, shewing the Value of Single and Joint Lives, &c. at different Rates of Interest. To which is added, a Method of Investigating the Values of Annuities by Approximation, without the Help of Tables. In 8vo. Price 3s. sew'd. 1742.

III. An Appendix to the Doctrine of Annuities, containing some Remarks on Mr. *Demoivre's* Book on the same Subject. In 8vo. Price 6d.

IV. A Treatise of Algebra; wherein the fundamental Principles are fully and clearly demonstrated, and applied to the Solution of a great Variety of Problems: To which is added, the Construction of a great Number of Geometrical Problems; with the Method of Resolving the same Numerically. In One Vol. 8vo. Price 6s.

V. The Elements of Plane Geometry; to which are added, an Essay on the *Maxima* and *Minima* of Geometrical Quantities, and a brief Treatise of Regular Solids; also the Mensuration of both Superficies and Solids: Together with the Construction of a large Variety of Geometrical Problems. In 8vo. Price 3s. 6d.

VI. Trigonometry Plane and Spherical, with the Construction and Application of Logarithms. 8vo. Price 1s. 6d.



THE
DOCTRINE and APPLICATION
OF
FLUXIONS.

PART the Second.

SECTION I.

*The Manner of investigating the FLUXIONS
of Exponentials, with Those of the Sides
and Angles of spherical Triangles.*

250. **T**HE Method of deriving the Fluxion of any Power, x^v , of a flowing Quantity; when the Exponent (v) is given or invariable, has been already shewn: But, if the Exponent be variable, that Method fails; in which Case the Quantity x^v is call'd an Exponential; whose Fluxion is thus determined.

Put $z = x^v$, and let the hyperbolic Logarithm of x be denoted by y ; then that of x^v (z) will, by the Nature of Logarithms, be $=vy$; and therefore its Fluxion $= vy + v\dot{y}$: But the Fluxion of the Logarithm of z ($=x^v$) is

T 2

* Art. 126. is also expressed by $\frac{\dot{z}}{z}$ *; whence we have $\frac{\dot{z}}{z} = \frac{\dot{v}}{v} + \frac{\dot{y}}{y}$,

and consequently $\dot{z} = z\frac{\dot{v}}{v} + y\frac{\dot{y}}{y}$: Which Equation, by substi-

† Art. 126. tuting $\frac{\dot{x}}{x}$ for its Equal $\frac{\dot{y}}{y}$ †, becomes $\dot{z} = z\frac{\dot{v}}{v} + \frac{zy\dot{x}}{x} =$

$$x^v y\dot{v} + x^v \times \frac{v\dot{x}}{x} = x^v y\dot{v} + vx^{v-1}\dot{x} = x^v \dot{v} \times \text{hyp. Log. } x$$

$$+ vx^{v-1}\dot{x}.$$

The same otherwise, without introducing the Properties of Logarithms.

251. Let $1+z=x$, and $n+v=w$, supposing n constant and w variable: Then $x^w = (1+z)^{n+v} = (1+z)^n$

$$\times (1+z)^v = (1+z)^n \times 1 + wz + \frac{w}{1} \times \frac{w-1}{2} \times z^2 +$$

* Art. 99. $\frac{w}{1} \times \frac{w-1}{2} \times \frac{w-2}{3} \times z^3 + \text{etc.} * = (1+z)^n \times$

$$1 + wz + \frac{1}{2}w^2 - \frac{1}{2}w \times z^2 + \frac{1}{6}w^3 - \frac{1}{2}w^2 + \frac{1}{3}w \times z^3 + \text{etc.}$$

whose Fluxion, found the common Way, is $n\dot{z} \times$

$$(1+z)^{n-1} \times 1 + wz + \frac{1}{2}w^2 - \frac{1}{2}w \times z^2 + \frac{1}{6}w^3 - \frac{1}{2}w^2 + \frac{1}{3}w$$

$$\times z^3 \text{ etc.} + (1+z)^n \times w\dot{z} + w\dot{z} + w\dot{v} - \frac{1}{2}\dot{v} \times z^2 + \frac{1}{2}w^2 - \frac{1}{2}w$$

$$\times 2z\dot{z} + \frac{1}{2}w^2\dot{v} - w\dot{v} + \frac{1}{3}\dot{v} \times z^3 + \frac{1}{6}w^3 - \frac{1}{2}w^2 + \frac{1}{3}w \times 3z^2\dot{z}$$

etc. which, by substituting $\dot{x}$ and $\dot{v}$ for their Equals $\dot{z}$

$$\text{and } \dot{v}, \text{ becomes } n\dot{x} \times (1+z)^{n-1} \times 1 + wz + \frac{1}{2}w^2 - \frac{1}{2}w$$

$$\times z^2 + \text{etc.} + (1+z)^n \times \dot{v}z + w\dot{x} + w\dot{v} - \frac{1}{2}\dot{v} \times z^2 + \text{etc.}$$

But, if w be, now, supposed to vanish, we shall have the true Value of the Fluxion when $v=n$; which, in

$$\text{that Circumstance, appears to be } = n\dot{x} \times (1+z)^{n-1}$$



$$+ \overline{1+z}^n \times \overline{z\dot{v} - \frac{1}{2}z^2\dot{v} + \frac{1}{3}z^3\dot{v} - \frac{1}{4}z^4\dot{v}} \text{ &c.} = v\dot{x} \times \overline{a}^{v-1} \\ + \dot{v}x^v \times \overline{z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4} \text{ &c.} \quad \text{Q. E. I.}$$

It is plain, because the Series, $z - \frac{1}{2}z^2 + \frac{1}{3}z^3 \text{ &c.}$ here brought out, is known to express the Fluent of $\frac{z}{1+z}$, or the hyperbolic Logarithm of $1+z^*$, that the * Art. 126.

two Conclusions agree exactly with each other: From either of which the following Rule, for the Fluxions of Exponentials, is deduced.

252. *To the Fluxion found by the common Rule (Art. 14.) considering the Exponent as constant, add the Quantity arising by multiplying the Fluxion of the Exponent, the hyperbolic Logarithm of the Root, and the proposed Quantity itself, continually, together: The Sum will be the Fluxion when the Exponent is variable.*

Thus, for Example, let the Quantity proposed be

$$\overline{a^2 + z^2}^z, \text{ then the Fluxion thereof will be } z \times 2z\dot{z} \times \overline{a^2 + z^2}^{z-1} + \dot{z} \times \overline{a^2 + z^2}^z \times \text{hyp. Log. } a^2 + z^2.$$

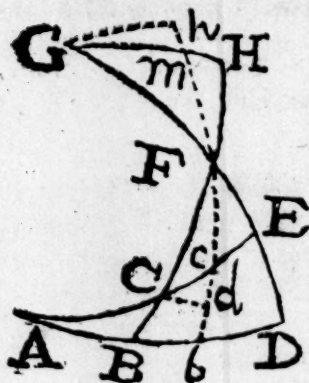
But, if the Root is constant, and only the Exponent variable, the Exponential will be more simple; and its Fluxion will then be had by barely multiplying the Quantity itself by the Product under the Logarithm of the Root and the Fluxion of the Exponent.

Thus, the Fluxion of a^x will be expressed by $a^x \times \dot{x}$

$\times \text{hyp. Log. } a$; and that of $\overline{a^2 + b^2}^{nx}$ by $\overline{a^2 + b^2}^{nx} \times nx \times \text{hyp. Log. } a^2 + b^2$. These Kind of Exponentials oftner occur, in Practice, than any other; but, as it is very rare that we meet with any, I shall therefore proceed now to the other Consideration proposed in the Head of this Section; namely, the Method of determining the Fluxions of the Sides and Angles of spherical Triangles (a Thing very useful in practical Astronomy) which I shall deliver in the following Propositions.

PROPOSITION I.

253. To determine the Ratio of the Fluxions of the several Parts of a right-angled spherical Triangle; supposing the Hypotenuse, one Leg, or one Angle, to remain constant, while the other Parts vary.



Let A, F, and G be the Poles of the three Great-Circles DEFG, ABD, and ACE; whereof the Position of each is supposed to continue invariable, while another Great-Circle HFCB is conceived to revolve about the Pole F: Whence, if GH be supposed perpendicular to FH, three variable right-angled Triangles, FGH, FCE, and ABC, will be

formed; in the first whereof, the Hypotenuse FG will remain constant; in the second, the Leg EF; and in the third the Angle A.

¶ Art. 134. Let $Bb (q)$ be the Fluxion (or indefinitely small Increment *) of the Base AB, or the Angle F; and let Cd meet the Great-Circle bFh , at Right-angles, in d ; then it will be (*per Spherics*) as $\text{Sin. FB (Rad.)} : \text{Sin. FC} :: Bb (q) : Cd = \frac{\text{Sin. FC}}{\text{Rad.}} \times q = \frac{\text{Co-f. BC}}{\text{Rad.}} \times q :$

$$\text{FC} :: Bb (q) : Cd = \frac{\text{Sin. FC}}{\text{Rad.}} \times q = \frac{\text{Co-f. BC}}{\text{Rad.}} \times q :$$

$$\text{And, Tang. C : Rad.} :: Cd \left(\frac{\text{Co-f. BC}}{\text{Rad.}} \times q \right) : \frac{\text{Co-f. BC}}{\text{Tang. C}} \times q = \text{the Fluxion of BC.}$$

$$\text{Moreover, Sin. C : Rad.} :: Cd \left(\frac{\text{Co-f. BC}}{\text{Rad.}} \times q \right) :$$

$$\frac{\text{Co-f. BC}}{\text{Sin. C.}} \times q = \text{the Fluxion of AC.}$$

. Lastly,

Lastly, *Sine* of FB (*Rad.*) : *Sin.* FH (BC) :: Bb (*q*) :

$\frac{\text{Sin. BC}}{\text{Rad.}} \times q (=Hm) = \text{the Fluxion of GH, or its Complement C.}$

Now, if the several Quantities, in these three Equations for the Triangle ABC, be expounded by their respective Equals in the other two Triangles CEF and FGH, we shall also have

$$\frac{\text{Sin. CF}}{\text{Tang. C}} \times q = - \text{Flux. CF.}$$

$$\frac{\text{Sin. CF}}{\text{Sin. C}} \times q = - \text{Flux. CE.}$$

$$\frac{\text{Co-f. CF}}{\text{Rad.}} \times q = \text{Flux. C.}$$

And

$$\frac{\text{Co-f. FH}}{\text{Co-tang. GH}} \times q = \text{Flux. FH.}$$

$$\frac{\text{Co-f. FH}}{\text{Co-f. GH}} \times q = \text{Flux. G.}$$

$$\frac{\text{Sin. FH}}{\text{Rad.}} \times q = - \text{Flux. GH.} \quad \text{Q. E. I.}$$

COROLLARY I.

254. Hence, if, in any right-angled Spherical-Triangle, the Hypothenufe be denoted by *h*, the two Legs by *L* and *l*, the Angles, respectively, adjacent to them by *A* and *a*, we shall, by substituting above, have three Equations for each of the three Cases. From the Comparison and Composition of which, the three following Tables are deduced; exhibiting all the different Varieties that can possibly happen, whether an Angle, a Leg, or the Hypothenufe be supposed invariable.

T 4

TABLE

TABLE I.

When one Angle A is invariable,

$$\begin{aligned}\dot{L} &= \frac{\text{Tang. } a}{\text{Co-f. } l} \times \dot{l} = \frac{\text{Sin. } a}{\text{Co-f. } l} \times \dot{b} = \frac{\text{Rad.}}{\text{Sin. } l} \times \dot{a} \\ \dot{l} &= \frac{\text{Co-f. } l}{\text{Tang. } a} \times \dot{L} = \frac{\text{Co-f. } a}{R} \times \dot{b} = \frac{\text{Co-tang. } l}{\text{Tang. } a} \times \dot{a} \\ \dot{b} &= \frac{\text{Co-f. } l}{\text{Sin. } a} \times \dot{L} = \frac{R}{\text{Co-f. } a} \times \dot{l} = \frac{\text{Co-tang. } l}{\text{Sin. } a} \times \dot{a} \\ \dot{a} &= \frac{\text{Sin. } l}{R} \times \dot{L} = \frac{\text{Tang. } a}{\text{Co-tang. } l} \times \dot{l} = \frac{\text{Sin. } a}{\text{Co-tang. } l} \times \dot{b}\end{aligned}$$

TABLE II.

When one Leg L is invariable,

$$\begin{aligned}\dot{A} &= \frac{\text{Tang. } a}{\text{Sin. } b} \times \dot{b} = \frac{\text{Sin. } a}{\text{Sin. } b} \times \dot{l} = -\frac{R}{\text{Co-f. } b} \times \dot{a} \\ \dot{a} &= -\frac{\text{Co-f. } b}{R} \times \dot{A} = -\frac{\text{Sin. } a}{\text{Tang. } b} \times \dot{l} = -\frac{\text{Tang. } a}{\text{Tang. } b} \times \dot{b} \\ \dot{b} &= \frac{\text{Sin. } b}{\text{Tang. } a} \times \dot{A} = \frac{\text{Co-f. } a}{R} \times \dot{l} = -\frac{\text{Tang. } b}{\text{Tang. } a} \times \dot{a} \\ \dot{l} &= \frac{\text{Sin. } b}{\text{Sin. } a} \times \dot{A} = \frac{R}{\text{Co-f. } a} \times \dot{b} = -\frac{\text{Tang. } b}{\text{Sin. } a} \times \dot{a}\end{aligned}$$

TABLE III.

When the Hyp. is invariable,

$$\begin{aligned}\dot{A} &= -\frac{\text{Co-tang. } l}{\text{Co-f. } L} \times \dot{L} = -\frac{\text{Co-f. } l}{\text{Co-f. } L} \times \dot{a} = \frac{R}{\text{Sin. } L} \times \dot{l} \\ \dot{L} &= -\frac{\text{Co-f. } L}{\text{Co-tang. } l} \times \dot{A} = \frac{\text{Sin. } l}{R} \times \dot{a} = -\frac{\text{Tang. } l}{\text{Tang. } L} \times \dot{l}\end{aligned}$$

Where, and also in the two preceding Tables, the Leg L is adjacent to the Angle A , and the Leg l to the Angle a .

COROLLARY II.

255. From the third original Equation, expressing the Fluxion of the Angle C (*Vid. Art. 253.*) it appears that the Superficies of any Spherical-Triangle ABC , is proportional to the Excess of its three Angles above two Right-Angles. For $(BCdb)$ the Fluxion of the Triangle ABC , is $= \text{Sine } BC \times Bb$, by *Art. 161.*) which

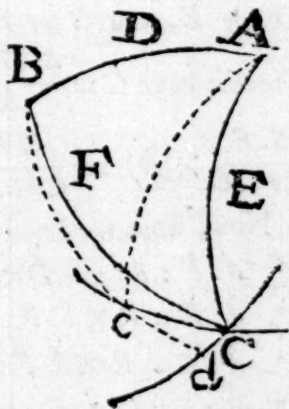
being to, $\frac{\text{Sin. } BC}{\text{Rad.}} \times Bb$, the Fluxion of the Angle C ,

above specified, in the constant Ratio of Radius to Unity, the Fluents themselves (properly corrected) must therefore be in that Ratio; that is, the Superficies of the Triangle ABC will always be proportional to the Increase of the Angle C , from its coinciding with A , or as the Excess of A and C above two Right-Angles.

PROPOSITION II.

256. To determine the Ratio of the Fluxions, or the indefinitely small Increments, of the different Parts of an oblique Spherical-Triangle ABC ; two Sides thereof AB , AC being invariable, in Length.

Let Cc be an indefinitely small Part of the Parallel described by the Extreme C of the given Side AC , in its Motion about the given Point A ; moreover, let Cd be Part of another Parallel, whose Pole is the given Point B ; let the Great-Circle Bc meet Cd in d ; and let the three Sides, AB , AC , and BC , of the Triangle be denoted by D , E , and F respectively.



Then,

Then (*per Spherics*) we shall have

$$R : S.E :: CAc (\dot{A}) : Cc = \frac{S.E}{R} \times \dot{A};$$

$$\text{And, } R : S.F :: CBd (\dot{B}) : Cd = \frac{S.F}{R} \times \dot{B}.$$

$$\text{Also, } R : S. dCc (ACB) :: Cc : \dot{F} = \frac{S.E \times S.C}{R^2} \times \dot{A};$$

$$\text{But } S.C : S.D :: S.B : S.E; \text{ therefore } S.E \times S.C \\ = S.D \times S.B, \text{ and consequently } \dot{F}, \text{ also, } = \frac{S.D \times S.B}{R^2} \\ \times \dot{A}.$$

$$\text{Again, } R : Co-f. dCc (ACB) :: Cc \left(\frac{S.E}{R} \times \dot{A} \right) : \\ \frac{S.E \times Co-f. C}{R^2} \times \dot{A} (=Cd) = \frac{S.F}{R} \times \dot{B};$$

$$\text{Whence } \dot{B} = \frac{S.E \times Co-f. C}{R \times S.F} \times \dot{A}.$$

$$\text{Lastly, } Co-t. cCd (C) : R :: Cd \left(\frac{S.F}{R} \times \dot{B} \right) : \dot{F} = \\ \frac{S.F}{Co-t. C} \times \dot{B}.$$

Whence, by the very same Argument (substituting *D* for *E*, and *C* for *B* in the two last Equations) we likewise have $\dot{C} = \frac{S.D \times Co-f. B}{R \times S.F} \times \dot{A}$, and $\dot{F} (= \frac{S.F}{Co-t. C} \times \dot{B}) = \frac{S.F}{Co-t. B} \times \dot{C}$.

Now, from the Equations thus found, it is manifest,

$$1^{\circ}. \dot{A} : \dot{F} :: R^2 : S.D \times S.B (:: Co-sec. D : S.B)$$

$$2^{\circ}. \dot{A} : \dot{B} :: R \times S.F : S.E \times Co-f. C$$

$$3^{\circ}. \dot{A} : \dot{C} :: R \times S.F :: S.D \times Co-f. B$$

$$4^{\circ}. \dot{B} : \dot{F} :: Co-t. C : S.F$$

$$5^{\circ}. \dot{C} : \dot{F} :: Co-t. B : S.F$$

$$6^{\circ}. \dot{B} : \dot{C} :: Co-t. C : Co-t. B (:: T. B : T. C) \quad Q. E. I.$$

257. These Proportions, for the Fluxions of the Parts of a Spherical-Triangle, are very useful in various Cases in *Practical Astronomy*; whereof I shall here put down one or two Instances.

The first is; To determine the annual Alteration of the Declination and Right-Ascension of a fixt Star, through the Precession of the Equinox.

Here *A* must denote the Pole of the Ecliptic, *B* that of the Equinoctial, and *C* the Place of the Star; and then (by the first and fourth Proportions) we have

$$\text{Co-sec. } D : \text{Sin. } B :: \dot{A} : \dot{F}; \text{ and}$$

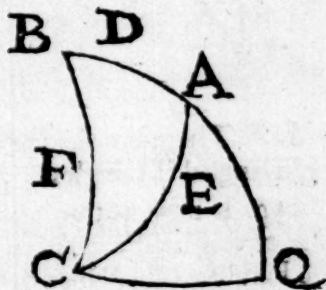
$$S. F : \text{Co-t. } C :: \dot{F} : \dot{B};$$

That is, 1°. As the Co-secant of the Obliquity of the Ecliptic is to the Sine of the Star's Right-Ascension from the *solstitial Colure*, so is the *Precession* of the Equinox, or Alteration of Longitude, to the Alteration of Declination.

2°. As the Co-sine of the Star's Declination is to the Co-tangent of its Angle of Position, so is the Alteration of Declination (found as above) to the Alteration of Right-Ascension corresponding.

The second Example is to find how much the Amplitude, and the Time of the apparent Rising and Setting of the Sun, or a Star, are affected by Refraction.

In this Case *A* must denote the Pole of the Equator, and *B* the *Zenith*, and the Side *BC* must be an Arch of 90 Degrees, so that the Star *C* may coincide with the Horizon *QC*: Then, from the very same Proportion, we have,



$$\text{Sin. } B : \text{Co-sec. } D :: \dot{F} : \dot{A},$$

$$\text{And, } R : \text{Co-t. } C :: \dot{F} : \dot{B}$$

$$\text{But, } R : \text{Co-t. } C (\text{T. } \angle CA) :: \text{Sin. } B (\text{C}\angle) : \text{Co-tang. } D (\text{Tang. } \angle A)$$

Hence

Hence it appears,

1°. That, as the Co-sine of the true Amplitude (considered independent of Refraction) is to the Tangent of the Pole's Elevation, so is the given horizontal Refraction to the Difference of Amplitudes thence arising.

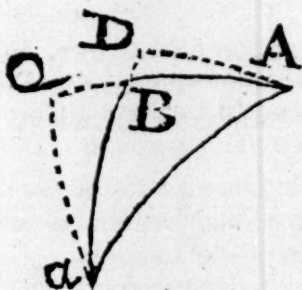
2°. And, that, as the Co-sine of the true Amplitude is to the Secant of the Pole's Elevation, so is the said horizontal Refraction to the Effect thereof in the Time of Rising, or Setting of the Sun, or Star.

But this last Proportion may be otherwise expressed, without the Amplitude: Thus,

$S. AB \times S. AC \times S. A : R^3 ::$ the horizontal Refraction, to the same Effect.

PROPOSITION III.

258. To determine the same as in the preceding Problem; supposing one Side AB and one of its adjacent Angles, B, to continue invariable.



If from the End of the given Side, opposite to the given Angle, a Perpendicular AD be let fall, that Perpendicular, as well as the Segment BD cut off thereby, will be a constant Quantity, while the other Parts of the Triangle AaD vary, by the Motion of *a* along the Arch

*a*BD. Therefore the Problem is resolved by *Case 2.* of right-angled Triangles. *Vid. Art. 254.*

259. It may not be amiss to give one Example of the Use of this last Proposition: Which shall be, in finding the Parallax of a Planet in Longitude and Latitude; that of Altitude being given.

Here *A* must stand for the Pole of the Ecliptic, *B* the Zenith, and *a* the Planet: Then, if the Hypothenuse *Aa* be denoted by *h*, the Leg. *Da* by *l*, and the given Parallax, in Altitude, by $\hat{l}$, it will appear, from the

the Place above quoted, that $\dot{A}$ (the Parallax in Long.) will be $= \frac{\sin. a}{\sin. b} \times \dot{i} = \frac{\sin. BaA}{\sin. Aa} \times \dot{i}$, and $\dot{b}$ (the Parallax in Lat.) $= \frac{\text{Co-f. } a}{\text{Rad.}} \times \dot{i} = \frac{\text{Co-f. } BaA}{\text{Rad.}} \times \dot{i}$.

If the Planet be in (or very near) the Ecliptic, and aQ be supposed a Portion of the Ecliptic, meeting AB , at Right-Angles, in Q , then (*per Spherics*) $\frac{\sin. BaA}{\sin. Aa}$
 $\left(\frac{\text{Co-f. } BaQ}{\text{Radius}} \right) = \frac{\text{Tang. } Qa}{\text{Tang. } Ba}$; also $\frac{\text{Co-f. } BaA}{\text{Rad.}} \left(\frac{\sin. BaQ}{\text{Rad.}} \right)$
 $= \frac{\sin. QB}{\sin. Ba}$; whence, by substituting these Values

above, we shall, in this Case, have $\dot{A} = \frac{\text{Tang. } Qa}{\text{Tang. } Ba} \times \dot{i}$ and $\dot{b} = \frac{\sin. QB}{\sin. Ba} \times \dot{i}$; that is, in Words,

As, the Tangent of the Planet's Zenith Distance, is to the Tangent of its Longitude from the nonagesimal Degree of the Ecliptic, so is the Parallax in Altitude to the Parallax in Longitude.

And, as the Sine of the Zenith Distance to the Co-sine of the Altitude [of the nonagesimal Degree, so is the Parallax in Altitude to the Parallax in Latitude.

Because the Parallax in Altitude, the horizontal Parallax (M) being given, is nearly $= \frac{\sin. Ba}{\text{Rad.}} \times M$, if

this Value be substituted for $\dot{i}$, in the two last Equations, we shall get $\dot{b} = \frac{\sin. QB}{\text{Rad.}} \times M$, and $\dot{A} = \frac{\text{Tang. } Qa \times \sin. Ba}{\text{Rad.} \times \text{Tang. } Ba}$

$\times M = \frac{\sin. AB \times \sin. BaA}{\text{Rad.}^2} \times M$.

Whence,

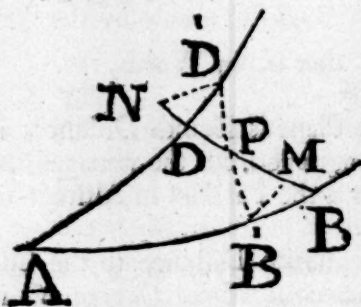
Whence, we have these two other Theorems, for finding the required Parallaxes immediately from the horizontal Parallax, without either the Altitude or its Parallax.

1. As Radius to the Co-sine of the Altitude of the *nonagesimal Degree* of the Ecliptic, so is the horizontal Parallax to the Parallax in Latitude.

2. And as the Square of Radius to the Rectangle under the Sines of the Altitude of the *nonagesimal Degree* and the Planet's Longitude from thence, so is the horizontal Parallax to the Parallax in Longitude.

PROPOSITION IV.

260. Still, to determine the same Thing; supposing, one Angle A, and the Length of its opposite Side BD (or $\dot{B}\dot{D}$) to remain constant.



Let $\dot{B}\dot{D}$ (equal to BD) intersect BD in an indefinitely small Angle at P, and meet AB

and AD in $\dot{B}$ and $\dot{D}$; also in BD produced let there be taken PN

$= PD$ and $PM = PB$,

and let N, $\dot{D}$, and M, $\dot{B}$ be joined.

Since, by Hypothesis, $DB = \dot{D}\dot{B} = MN$, if from the first and last of these equal Quantities DM, common, be taken away, there will remain $BM = DN$.

Moreover, since the Triangles $BM\dot{B}$ and $DN\dot{D}$, in their ultimate State, may be considered as rectilineal,
 • Art. 134. and right-angled at M and N*, it will therefore be, as

$$BM : B\dot{B} :: \text{Co-f. } B : \text{Radius}$$

$$\text{And } DN : D\dot{D} :: \text{Co-f. } D :: \text{Radius.}$$

From

From whence, the Extremes in both Proportions being the same, we have $B\dot{B} : D\dot{D} :: Co-f. D : Co-f. B$: And therefore, if AB be denoted by H and AD by K , it appears that $\dot{H} : \dot{K} :: Co-f. D : Co-f. B$.

Again, *per Spherics*, $Sin. A : Sin. BD (G) :: Sin. D : Sin. H :: Flux. Sin. D : Flux. Sin. H$; because, the Sines themselves being in a constant Ratio, their Fluxions must be in the same Ratio: But the Fluxion of the Sine of any Arc, or Angle, is to the Fluxion of the Arc or Angle itself, as the Co-sine to Radius *; * Art. 142.

Therefore the *Flux. Sin. D* being $= \frac{Co-f. D}{Rad.} \times \dot{D}$, and

Flux. Sin. H $= \frac{Co-f. H}{Rad.} \times \dot{H}$, it follows that, $Sin. A$

$: Sin. G :: Co-f. D \times \dot{D} : Co-f. H \times \dot{H}$; or $\dot{D} : \dot{H} :: Sin. A \times Co-f. H : Sin. G \times Co-f. D$: And, by the very same Argument, $\dot{B} : \dot{K} :: Sin. A \times Co-f. K : Sin. G \times Co-f. B$. Now, by compounding the former of these two Proportions with the first above given, we get, $\dot{D} : \dot{K} :: Sin. A \times Co-f. H : Sin. G \times Co-f. B$. And, by compounding this last with $\dot{K} : \dot{B} :: Sin. G \times Co-f. B : Sin. A \times Co-f. K$ (that immediately preceding it) we also obtain $\dot{D} : \dot{B} :: Co-f. H : Co-f. K$.

Whence, by collecting these several Proportions together, we have the following Table, for all the different Cases.

$$\dot{H} : \dot{K} :: Co-f. D : Co-f. B$$

$$\dot{D} : \dot{B} :: Co-f. H : Co-f. K$$

$$\dot{D} : \dot{H} :: Tang. D : Tang. H$$

$$\dot{B} : \dot{K} :: Tang. B : Tang. K$$

$$\dot{K} : \dot{D} :: Sin. G \times Co-f. B : Sin. A \times Co-f. H$$

$$\dot{K} : \dot{B} :: Sin. G \times Co-f. D : Sin. A \times Co-f. K$$

It

It may be observed, that the fourth and the last are no new Cases, but only the third and fifth repeated: And that, though the former of the two, last named, differs from that found above; yet it is very easily deduced

from it: For, since it appears that $\dot{D} : \dot{H} :: \frac{\text{Sin. } A}{\text{Co-f. } D} :$

$\frac{\text{Sin. } G}{\text{Co-f. } H}$, and because $\text{Sin. } A : \text{Sin. } G :: \text{Sin. } D : \text{Sin. } H$,

it follows that $\dot{D} : \dot{H} :: \frac{\text{Sin. } D}{\text{Co-f. } D} : \frac{\text{Sin. } H}{\text{Co-f. } H} ::$

$\text{Tang. } D : \text{Tang. } H$.

Q. E. I.

There is yet another Problem, when two Angles remain constant; but this, by taking the Triangle formed by the Poles of the three given Circles, is reduced to Problem 2.

SECTION II.

Of the Resolution of fluxional Equations, or the Manner of finding the Relation of the flowing Quantities from that of the Fluxions.

261. **W**HEN an Equation, expressing the Relation of the Fluxions of the two variable Quantities, contains *only* one of those Fluxions with its respective flowing Quantity in each Term, the Relation of the Quantities will be obtained by finding the Fluent of every Term; as has been already taught, in *Sect. VI. Part I.*

Thus, if $ax^2\dot{x} = y^3\dot{y}$, then will $\frac{ax^3}{3} = \frac{y^4}{4}$.

And, if $x^n y^m \dot{x} = ay$; by reducing it first to $x^n \dot{x} = ay^{-m}y$ (so that its variable Quantities may be separated) we have $\frac{x^{n+1}}{n+1} = \frac{ay}{1-m}$.

But,

But, if the given Equation has its indeterminate Quantities and their Fluxions so complicated together, that it cannot be brought under the Form there prescribed, the Task will become much more difficult; nor is there any general Method to be given for such Kinds of Equations, whereof there are an infinite Variety.

The Method of Infinite Serieses (in some measure explained already, and more fully considered hereafter) is indeed very comprehensive, and may be apply'd to good Purpose in various Cases; but, being tedious and attended with a Number of Inconveniencies, it is a Method we ought never to have Recourse to till we have try'd what may be, otherways, effected, by help of such particular Rules and Observations as we have been able to collect.

Accordingly, I shall, here, first point out some of the most proper Ways to be tried, in order, if possible, to bring out the Solution without an Infinite Series.

262. *The first Method is, by multiplying, or dividing, the given Equation into some Power or Product of the Quantities concerned; so as to bring it, if possible, under the Form of such Fluxions, as, we know, do arise, if not from the first, yet from the second, or third, of the three general Rules in the direct Method.*

Thus, if the given Equation be $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$;

then, the whole being multiply'd by xy , so that the two first Terms, $y\dot{x} + x\dot{y}$, may become the (known) Fluxion of

the Rectangle xy *, there arises $y\dot{x} + x\dot{y} = \frac{x^{m+1} \dot{x}}{ay^{n-1}}$: But * Art. 18.

still we are at a Loss for the Fluent of the last Term, unless n be taken $= 1$ (so that y may vanish.) In that

Case we have $xy = \frac{x^{m+2}}{m+2 \times a}$; expressing the Relation

of the Fluents when that of the Fluxions is $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} =$

$\frac{x^m \dot{x}}{ay}$: Which appears to be the only Case, of the given

Equation, where this Method is of Use:

U

Again

Again, let the Equation $\frac{p\dot{x}}{x} + \frac{r\dot{y}}{y} = \frac{x^m \dot{x}}{ay^n}$ be proposed.

Here, multiplying by $x^p y^r$ (where the Exponents are the same as the Coefficients of $\frac{\dot{x}}{x}$ and $\frac{\dot{y}}{y}$) we get

$$px^{p-1}\dot{x}y^r + x^p \times ry^{r-1}\dot{y} = \frac{x^{m+p}\dot{x}}{ay^{n-r}}; \text{ in which the}$$

former Part of the Equation is known to express the

* Art. 15. Fluxion of $x^p y^r$ *. Therefore, when $n=r$, the Relation of the Fluents may be found, and will be expressed by

$$x^p y^r = \frac{x^{m+p+1}}{m+p+1 \times a} : \text{ Which, if no Correction by a}$$

constant Quantity be necessary, may be reduced to

$$y^r = \frac{x^{m+1}}{m+p+1 \times a}.$$

† The same Method may also be extended to Fluxions of the higher Orders: Let $\ddot{x} - x\dot{z}^2 = f\dot{z}^2$ (which Equation occurs hereafter, in the Resolution of a Problem of some Difficulty.) Then, multiplying by $\dot{x}$, it becomes $\dot{x}\ddot{x} - x\dot{x}\dot{z}^2 = f\dot{z}^2\dot{x}$; where, $\dot{z}$ being constant, each Term admits, now, of a perfect Fluent, and we therefore

have $\frac{\dot{x}^2}{2} - \frac{x^2\dot{z}^2}{2} = f\dot{z}^2x$: From whence, supposing no

Correction necessary, $\dot{z} = \frac{\dot{x}}{\sqrt{2fx + xx}}$, and $z = \text{hyp.}$

Log. $f + x + \sqrt{2fx + x^2}$ (by Art. 126.)

263. It may happen that the Solution of an Equation will become more easy by first taking the Fluxion thereof; when, by that means, some of the Terms destroy each other.

The following is an Instance of it (which, also, occurs hereafter) Let $y + \frac{y \times a - x}{\dot{x}} = x - \frac{y\dot{x}}{\dot{y}}$: Whose Fluxion,

ion,

ion, making $\dot{x}$ constant, is $\dot{y} + \frac{\ddot{y} \times \overline{a-x} - \dot{x}\dot{y}}{\dot{x}} = \dot{x} -$

$\frac{\dot{y}\dot{x}\dot{y} - y\dot{x}\dot{y}}{\dot{y}\dot{y}}$: Which, by reason of the Terms destroying

one another, is reduced to $\frac{\ddot{y} \times \overline{a-x}}{\dot{x}} = \frac{y\dot{x}\dot{y}}{\dot{y}\dot{y}}$: Therefore,

by expunging $\ddot{y}$, &c. we get $\dot{y}\dot{y}^{-\frac{1}{2}} = \dot{x} \times \overline{a-x}^{-\frac{1}{2}}$, and consequently $2y^{\frac{1}{2}} = -2 \times \overline{a-x}^{\frac{1}{2}} + \text{some constant Quantity}$.

264. Another Method, applicable to Equations, of the first Order of Fluxions, wherein only one of the two variable Quantities (x or y) enters, is, to substitute for the Ratio of the two Fluxions ($\dot{x}$ and $\dot{y}$): From whence the Value of that Quantity will be had, immediately, in Terms of the said assumed Ratio: And then, by taking its Fluxion, that of the other Quantity (and from thence the Quantity itself) will become known.

Thus, let $ax\dot{y}^3 = y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^2$ (being the Equation of the Curve that generates the Solid of the least Resistance, when the Bulk and greatest Diameter are given.)

Then, by putting $\frac{\dot{x}}{\dot{y}} = v$, and substituting above, we

get $av\dot{y}^4 = y \times \overline{v^2\dot{y}^2 + \dot{y}^2}^2 = y\dot{y}^4 \times \overline{v^2 + 1}^2$; and con-

sequently $y = \frac{av}{\overline{v^2 + 1}^2}$: Therefore $\dot{y} = \frac{av\dot{v} - 3av^2\dot{v}}{\overline{vv + 1}^3}$;

and consequently $\dot{x} (= v\dot{y}) = \frac{av\dot{v} - 3av^3\dot{v}}{\overline{vv + 1}^3}$: Whose

Fluent may be found, from Art. 84. or, otherwise, thus: Put $w^2 = v^2 + 1$; then $v^2 = w^2 - 1$, and $w\dot{w} = v\dot{v}$; by substituting which Values there arises $\dot{x} =$

$\frac{aw\dot{w} - 3aw\dot{w} \times \overline{w^2 - 1}}{w^6} = 4a\dot{w}w^{-5} - 3a\dot{w}w^{-3}$; and

$$\begin{aligned} \text{fore } x &= \frac{4aw^{-4}}{-4} - \frac{3aw^{-2}}{-2} = -\frac{a}{w^4} + \frac{3a}{2w^2} = \frac{3aw^2 - 2a}{2w^4} \\ &= \frac{3a \times \overline{v^2 + 1} - 2a}{2 \times \overline{vv + 1}^2} = \frac{a \times \overline{3vv + 1}}{2 \times \overline{vv + 1}^2}; \text{ which, corrected} \\ & \text{(by taking } y, \text{ or } v=0) \text{ becomes } x = \frac{a \times \overline{3vv + 1}}{2 \times \overline{vv + 1}^2} - \frac{a}{2}. \end{aligned}$$

From this Equation, by completing the Square, &c. v may be found in Terms of x ; whence the corresponding Value of y ($= \frac{av}{\overline{vv + 1}^2}$) will also be known.

265. *The fourth Method*, which chiefly obtains when one of the indeterminate Quantities and its Fluxion, arise but to a single Dimension each, may be thus :

Let the Value of that Quantity, which is least involved, be first sought, from the fictitious Equation arising by neglecting all the Terms in the given Equation, where neither that Quantity, nor its Fluxion, are found : Then, to that Value, let some Power, or Powers, of the other Quantity, with unknown Coefficients, be added (according to the Dimensions of the Terms neglected) and let the Sum be substituted in the given Equation, as the true Value of the first mentioned Quantity : By which means a new Equation will result ; from whence the assumed Coefficients may, sometimes, be determined.

Ex. Let the given Equation be $cx^2\dot{x} + y\dot{x} = ay$.

By neglecting $cx^2\dot{x}$, or feigning $y\dot{x} = ay$, we get

$$\frac{\dot{x}}{a} = \frac{\dot{y}}{y} : \text{ and consequently } \frac{x}{a} = \text{hyp. Log. } y - \text{hyp.}$$

* Art. 126. and 78. $\text{Log. } d^* = \text{hyp. Log. } \frac{y}{d}$; d being any constant Quantity, which the Nature of the Problem may require.

Hence $\frac{y}{d} =$ the Number whose hyperbolical Logarithm

is $\frac{x}{a}$: Which Number, if M be put for (2,71828 &c.)

the Number whose hyp. Log. is Unity, will be ex-

pressed by $\overline{M}^{\frac{x}{a}}$ (since it is evident that the hyp. Log. hereof is $\frac{x}{a} \times \text{Log. } M = \frac{x}{a}$): Therefore $\frac{y}{d} =$

$\overline{M}^{\frac{x}{a}}$ and $y = d \times \overline{M}^{\frac{x}{a}}$. Now, to the Value thus found, let there be added Ax^2+Bx+C , in order to get the true Value; and then, y being $= 2Axx+Bx+\frac{dx}{a}$

$\times \overline{M}^{\frac{x}{a}}$, we shall, by substituting in the given Equat. Art. 14.

tion, have $cx^2x+Ax^2x+Bxx+Cx+dx\overline{M}^{\frac{x}{a}} = 2Aaxx$

$+Bax+dx\overline{M}^{\frac{x}{a}}$, and consequently $c+A \times x^2x + B-2Aa \times xx+C-Ba \times x=0$. Whence $A=-c$, Art. 84.
 $B=-2ac$, $C=-2aac$; and consequently $y=-c \times$

$x^2+2ax+2aa+dx\overline{M}^{\frac{x}{a}}$. By the very same Way, the Value of y , in the Equation $cx^n+yx=ay$, will come

out $= -c \times x^n+ax^{n-1}+n.n-1.a^2x^{n-2}+n.n-1.$

$n-2.a^3x^{n-3}+Ec. + d\overline{M}^{\frac{x}{a}}$.

266. But, what is a little remarkable, in these Equat-

tions, is, that the *Exponential* $d\overline{M}^{\frac{x}{a}}$, tho' a variable Quantity, should only serve, as it were, to correct the Fluent, or perform the Office of a constant Quantity. What I here mean will plainly appear, if it be considered, that the Equation $y=-c \times x^2+2ax+2aa$, where the said *Exponential* is wanting, answers all the Conditions of the fluxional Equation first proposed; which, upon Trial, will be found; and must needs be

the Case, seeing d may be, either, taken Nothing at all, or any Quantity at Pleasure.

But the Equation $y = -c \times \overline{x^2 + 2ax + 2a^2}$ (when $\frac{x}{dM^a}$ is wanting) cannot be corrected, in the usual Way, so as to give $y=0$, when $x=0$; since, if any other constant Quantity, besides $-2a^2c$ be introduced, the first Conditions will not be answer'd: The Correction must, therefore, be by the Exponential dM^a ; and is thus.

Since $y = -cx^2 - 2cax - 2ca^2 + dM^a$, if y be taken $=0$ and $x=0$, then $-2ca^2 + dM^0 = 0$, or $d = 2ca^2$; and so the Equation, truly corrected, is $y = -c \times \overline{x^2 + 2ax + 2a^2 + 2a^2cM^a}$.

267. We come now to the last Method; namely, that of Infinite Series; which, tho' less accurate, is vastly more comprehensive, than any yet explained: The Manner of it is thus:

For the Quantity whose Value you would find, let an Infinite Series, consisting of the Powers of the other Quantity with unknown Coefficients, be assumed; which Series, together with its Fluxion, or Fluxions, must be substituted instead of their Equals in the given Equation; whence a new Equation will arise, from which, by comparing the homologous Terms, the assumed Coefficients, and consequently the Value sought, will be determined.

Thus, let the Equation $\frac{\dot{x}}{1+x} = y$ (reducible to $\dot{x} - y - xy = 0$) be proposed; to find x in Terms of y . Then, assuming $x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 \&c.$ We have $\dot{x} = Ay + 2Byy + 3Cy^2y + 4Dy^3y + 5Ey^4y + \&c.$ Which Values being substituted in $\dot{x} - y - xy = 0$, we get $Ay + 2Byy + 3Cy^2y + 4Dy^3y + \&c. - y - Ay - By^2y - Cy^3y - \&c. = 0$.

There-

Therefore $A - 1 = 0$, or $A = 1$; $2B - A = 0$, or $B = \frac{A}{2} = \frac{1}{2}$; $3C - B = 0$, or $C = \frac{B}{3} = \frac{1}{2 \cdot 3}$; $4D - C = 0$, or $D = \frac{C}{4} = \frac{1}{2 \cdot 3 \cdot 4} \mathcal{E}c$.

And consequently $x (Ay + By^2 + Cy^3 \mathcal{E}c) = y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} + \frac{y^5}{2 \cdot 3 \cdot 4 \cdot 5} + \mathcal{E}c$.

Again, let it be required to find the Value of y , in the Equation $cx^2\dot{x} + y\dot{x} = ay$, or $ay - y\dot{x} - cx^2\dot{x} = 0$. Here, assuming $y = Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5 + Fx^6 \mathcal{E}c$. and proceeding as before, we shall have

$$\left. \begin{array}{l} aA\dot{x} + 2aBx\dot{x} + 3aCx^2\dot{x} + 4aDx^3\dot{x} + 5aEx^4\dot{x} + \mathcal{E}c \\ 0 - Ax\dot{x} - Bx^2\dot{x} - Cx^3\dot{x} - Dx^4\dot{x} - \mathcal{E}c \\ 0 \quad 0 \quad - cx^2\dot{x} \end{array} \right\} \parallel 0$$

Whence $A = 0$; $2aB = A = 0$; $3aC = B + c = c$, or

$$C = \frac{c}{3a}; 4aD = C = \frac{c}{3a}, \text{ or } D = \frac{c}{3 \cdot 4a^2}; 5aE = D$$

$$= \frac{c}{3 \cdot 4a^2}, \text{ or } E = \frac{c}{3 \cdot 4 \cdot 5a^3} \mathcal{E}c. \text{ and consequently } y$$

$$(Ax + Bx^2 + Cx^3 + \mathcal{E}c) = \frac{cx^3}{3a} + \frac{cx^4}{3 \cdot 4a^2} + \frac{cx^5}{3 \cdot 4 \cdot 5a^3}$$

$$+ \frac{cx^6}{3 \cdot 4 \cdot 5 \cdot 6a^4} + \mathcal{E}c.$$

268. It appears from this Example, that the Quantity to be found, will not always require all the Terms of the Series $Ax + Bx^2 + Cx^3 \mathcal{E}c$. And it may happen, in innumerable Cases, that the Series to be assumed will demand a very different Law from *that* where the Exponents proceed according to the Terms of an arithmetical Progression having Unity for the common Difference. And, indeed, the greatest Difficulty we have here to encounter, is, to know what Kind of Series, with regard to its Exponents, ought to be assumed, so as to answer the Conditions of the Equation, without introducing more Terms than are actually necessary.

The following Rules will be found very useful upon this Occasion: Which, though they may become impracticable in certain particular Cases, never take in any superfluous Terms.

1°. Having (if necessary) freed your Equation from Fractions and Surds, let the Quantity, whose Value is sought, be supposed equal to some Power of the other Quantity with an unknown Exponent (n); and let that Power, together with its Fluxion, or Fluxions, be substituted for their (supposed) Equals in the given Equation.

2°. Let the least Exponents of the variable, or indeterminate, Quantity, in the new Equation, thence arising, be put equal to each other: Whence the Value of the unknown Exponent n will be found.

3°. Substitute the Value of n , so found, in all the Exponents where n is concerned; and then take the Difference between one of the equal ones, above mention'd, and every other Exponent, of the variable Quantity, in the whole Equation.

4°. To these Differences, write down all the least Numbers that can be compos'd out of them, by continual Addition, either to themselves, or to one another; till you have, by that means, got, in the whole, as many different Terms, as you would have the required Series continued to.

5°. Lastly, let each of those Terms be increased by the Value of n (found by Rule 2.) and you will then have the Exponents of the Series to be assumed.

EXAMPLE I.

269. Let the Value of x , in the Equation $a^2 \dot{x}^2 + x^2 \ddot{x}^2 - a^2 \ddot{x}^2 = 0$, be required.

First, by writing z^n for x , and $nz^{n-1}\dot{z}$ for $\dot{x}$, the Indices of z will be $2n-2$, $2n$, and 0 (which are determined by Inspection, without regarding the Coefficients) whereof the two least ($2n-2$ and 0) being put equal to each other, we here find $n=1$: Therefore, the Exponents being 0 , 2 , 0 , the Differences (according to Rule 3.) are also 0 , 2 ; from whence, by adding 2 continually, we get 0 , 2 , 4 , 6 , 8 &c. which (being each in-

increased by the Value of n give 1, 3, 5, 7, 9 &c. for the Exponents in this Case.

Let, therefore, $x = Az + Bz^3 + Cz^5 + Dz^7 + \&c.$ Then, putting $z = 1$, in order to facilitate the Operation, we shall have $x = A + 3Bz^2 + 5Cz^4 + 7Dz^6 + \&c.$ which two Values being squared, and substituted in the given Equation, it will become

$$\left. \begin{array}{l} a^2 A^2 + 6a^2 ABz^2 + 10a^2 ACz^4 + 14a^2 ADz^6 + \&c. \\ \quad + 9a^2 B^2 z^4 + 30a^2 BCz^6 + \&c. \\ * \quad + A^2 z^2 \quad + 2ABz^4 + 2ACz^6 + \&c. \\ - a^2 \quad \quad \quad + B^2 z^6 + \&c. \end{array} \right\} \parallel 0$$

Whence, $a^2 A^2 = a^2$, and therefore $A = 1$; $6a^2 B = -$

$$A, \text{ and therefore } B = -\frac{1}{6a^2} = -\frac{1}{2 \cdot 3a^2}; \quad 10a^2 AC$$

$$= -9a^2 B^2 - 2AB = -B \times \overline{9a^2 B + 2A} = -B \times$$

$$-\frac{3}{2} + 2 = -\frac{B}{2} = \frac{1}{2 \cdot 3 \cdot 2a^2}, \text{ and therefore } C =$$

$$\frac{1}{2 \cdot 3 \cdot 4 \cdot 5a^4}; \quad 14AD = -30a^2 \times -\frac{1}{6a^2} \times \frac{1}{120a^4} -$$

$$2 \times \frac{1}{120a^4} - \frac{1}{36a^4} = \frac{1}{24a^4} - \frac{1}{60a^4} - \frac{1}{36a^4} = -$$

$$\frac{1}{360a^4}, \text{ and therefore } D = -\frac{1}{14 \cdot 360a^6} = -$$

$$\frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7a^6}; \text{ and, consequently, } x = z - \frac{z^3}{2 \cdot 3a^2} +$$

$$\frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5a^4} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7a^6} \&c.$$

EXAMPLE II.

270. Let the given Equation be $a^2 xy - 2a^2 xy + axx^2 + x^3 y = 0$; to find y .

Here, substituting x^* for y , the Exponents will be $n-1$, $n-1$, 1, and $n+1$; where, making $n-1=1$,
we

we get $n=2$: Whence, the Differences being 0, 2, the Series to be assumed for y will be $Ax^2 + Bx^4 + Cx^6 + Dx^8 + Ex^{10} + \mathcal{E}c$. From which, making $x=1$, we have $y=2Ax+4Bx^3+6Cx^5+8Dx^7+\mathcal{E}c$. and

$$\ddot{y}=2A+12Bx^2+30Cx^4+56Dx^6$$

And, these Values being substituted, the Equation becomes

$$\left. \begin{aligned} 2A^2Ax + 12a^2Bx^3 + 30a^2Cx^5 + 56a^2Dx^7 + \mathcal{E}c. \\ -4a^2Ax - 8a^2Bx^3 - 12a^2Cx^5 - 16a^2Dx^7 + \mathcal{E}c. \\ +ax + 2Ax^3 + 12Bx^5 + 30Cx^7 + \mathcal{E}c. \end{aligned} \right\} = 0$$

$$\text{Therefore } A = -\frac{1}{2a}; B = -\frac{2A}{4a^2} = -\frac{1}{4a^3};$$

$$C = -\frac{12B}{18a^2} = \frac{1}{6a^5}; D = -\frac{30C}{40a^2} = -\frac{1}{8a^7} \mathcal{E}c.$$

$$\text{and so } y = \frac{x^2}{2a} - \frac{x^4}{4a^3} + \frac{x^6}{6a^5} - \frac{x^8}{8a^7} + \frac{x^{10}}{10a^9} - \mathcal{E}c.$$

Which Series is known to express the Fluent of $\frac{axx}{a^2+x^2}$,

or, $\frac{1}{2}a \times \text{hyp. Log. } \frac{a^2+x^2}{aa}$: Consequently y is also =

$\frac{1}{2}a \times \text{hyp. Log. } \frac{a^2+x^2}{a^2}$. In this manner, it comes to

pass, that, though we are obliged, in very complicated Cases, to have recourse to Infinite Serieses, we are sometimes able, at last, to give the Solution in finite Terms, or, at least, by help of Logarithms, Sines and Tangents: Which will always happen when the Series can be summed, or is found to agree with that arising from some known Quantity.

271. Sometimes it happens, in Equations involving the higher Orders of Fluxions, that the Exponents, mention'd in *Rule 2.* whereof the least ought to be made equal to each other, are so expressed, as to render such an Equality impossible. When this is the Case, the Value of n , and the first Term of the required Series, can only be determined from the Nature of the Problem to which the Equation belongs. We know,
in-

indeed, from the Equation itself, that n must be either equal to Nothing, or to some positive Integer, less than that expressing the Order of the highest Fluxion in the Equation: Because the Term that has the least Exponent, and which therefore cannot be compared with any other (being always affected by two or more of the Factors $n, n-1, n-2, \&c.$) will then (one of those Factors being $=0$) vanish intirely out of the Equation; which, thereby, is render'd possible.

When n and A are known, the rest of the Terms will be found in the common Way, as in

EXAMPLE III.

Where the Equation proposed is $y\dot{x}^2 + a\dot{y} - a^2\ddot{y} = 0$;
to find y .

By supposing $\dot{x}=1$, and writing x^n for y , nx^{n-1} for $\dot{y}$, and $n \times n-1 \times x^{n-2}$ for $\ddot{y}$, we get $x^n + nax^{n-1} - n \times n-1 \times a^2 x^{n-2}$: But it is plain that no two of the Indices of x can, *here*, be equal: The Value of n must therefore be either $=0$, or Unity (in both which Cases the Term $-n \times n-1 \times a^2 x^{n-2}$ vanishes) but I shall take the latter Value, and suppose the first Term of the Series to be Ax ; then, the Differences of the foresaid Exponents being 1 and 2, the Law of the Series will be expressed by 1, 2, 3, 4 $\&c.$ Whence, assuming $y = Ax + Bx^2 + Cx^3 + Dx^4 \&c.$ and proceeding as in the former Examples, y will be found $= A$ into $x + \frac{x^2}{2a} + \frac{x^3}{3a^2} + \frac{x^4}{8a^3} + \frac{x^5}{24a^4} + \frac{x^6}{90a^5} \&c.$ or $= A$ into $x + \frac{x^2}{2a} + \frac{2x^3}{2.3a^2} + \frac{3x^4}{2.3.4a^3} + \frac{5x^5}{2.3.4.5a^4} + \frac{8x^6}{2.3.4.5.6a^5} + \&c.$ where the Law of Continuation is manifest, the Coefficient of every Numerator being composed by the Addition of the two preceding ones.

272. It

272. It will be proper to observe here, that, in Equations like the two last proposed, where the higher Orders of Fluxions are concerned, the Series expressing the Relation of the two Quantities must always be found in Terms of the Quantity flowing uniformly. And, that, if the Number of Dimensions of the Fluxion of the said Quantity, after Substitution, be not the same in every Term, the Equation itself, put down to be resolved, is absurd and impossible, and such as never can arise in the Solution of any Problem. In all proper Equations the Number of fluxional Points (supposing the Powers of the Fluxions to be wrote without Indices) will be the same in every Term.

EXAMPLE IV.

273. Where let the given Equation be $a^3y - ay^2x + x^2yy = x^3x$; to find y .

By proceeding as usual the Indices will here be $n-1$, $2n$, $2n+1$ and 3 ; whereof the least (which can be no other than $n-1$ and 3) being compared, n will be given $=4$: And the Differences will therefore be $0, 5, 6$; to which the Double of the Second and the Sum of the second and third, &c. being put down, and then every Term increased by 4 , there arises $4, 9, 10, 14, 15, 16, 19$ &c. for the Exponents of the Series to be assumed for y .

Let therefore $y = Ax^4 + Bx^9 + Cx^{10} + Dx^{14}$ &c. then, making $x=1$, y is $= 4Ax^3 + 9Bx^8 + 10Cx^9 + 14Dx^{13} + \text{&c.}$

And, by substituting these Values above, we have

$$\left. \begin{aligned} 4a^3Ax^3 + 9a^3Bx^8 + 10a^3Cx^9 + 14a^3Dx^{13} + \text{&c.} \\ -x^3 - aA^2x^8 + 4A^2x^9 - 2aABx^{13} + \text{&c.} \end{aligned} \right\} = 0$$

Whence $A = \frac{1}{4a^3}$, $B = \frac{1}{144a^8}$ &c.

And $* y = \frac{x^4}{4a^3} - \frac{x^9}{144a^8} - \frac{x^{10}}{40a^9} + \frac{x^{14}}{4032a^{13}}$ &c.

** If, for y , the Series $Ax^4 + Bx^9 + Cx^{10} + Dx^{14}$ &c. whose Exponents are in arithmetical Progression, had been assumed, according to the Method of some very good Authors, no less than seven superfluous Terms must have been introduced to obtain the four above given.*

274. Before I quit this Subject, it may not be amiss to subjoin the following Remarks.

1°. If the indeterminate Quantities are great in respect to the given ones, a descending Series will, in most Cases (where it is practicable) converge better than an ascending one. To obtain such a Series, compare the greatest Exponents, mention'd in *Rule 2.* instead of the least, and proceed according to the third and fourth Rules *, whence a Series of Numbers will be found; * Art. 262. which, being successively subtracted from the Value of n , you will have the Exponents of a descending Series.

Thus, let the common-algebraic Equation $a^3x+ax^3-a^3y-y^4=0$ be propounded; to find y , when x is great in comparison of a .

Then, proceeding as usual, the Exponents of the four Terms of the Equation will be 1, 3, n , $4n$; whereof the two greatest ($4n$ and 3) being made equal, we get $n=\frac{3}{4}$; therefore the Differences are 0, 2 and $2\frac{1}{4}$; and the Numbers to be subtracted from n , are 0, 2, $\frac{3}{4}$, 4, $\frac{17}{4}$, &c. Consequently the Series to be assumed for y is

$Ax^{\frac{3}{4}}+Bx^{-\frac{5}{4}}+Cx^{-\frac{6}{4}}+Dx^{-\frac{13}{4}}+\&c.$ From whence

$$y \text{ will be found } = a^{\frac{1}{4}}x^{\frac{3}{4}} + \frac{a^{\frac{9}{4}}}{4x^{\frac{5}{4}}} - \frac{a^{\frac{10}{4}}}{4x^{\frac{6}{4}}} - \frac{3a^{\frac{17}{4}}}{32x^{\frac{13}{4}}} \&c.$$

2°. But, if the Quantity (x) in whose Terms the other is to be expressed, be neither much greater nor much smaller than the given Quantity (a), it will be proper to substitute for the Excess, or Defect, of the said Quantity (x) above, or below, some given Quantity; so that, having, by this means, exterminated x , the Series arising from the new Equation (wherein the said Excess, or Defect, is the converging Quantity) will have a due Rate of Convergency.

The Use of this is so obvious that it needs no Example, or farther Explanation.

3°. Lastly, it will be proper to observe, that, if the Equation for the Value of A , arising from the first Column of homologous Terms, admits of two or more equal

equal Roots (which is a Case that may, perhaps, never happen in Practice) all the foregoing Precepts will be insufficient; unless the Equation also admits of some other Root, besides the equal ones, whereby A may be more commodiously expressed. To determine the Exponents, in that particular Case, divide each of the Differences mention'd in *Rule 3.* by the Number of the equal Roots; and then proceed as usual. The Reasons of which, as well as of the Rules themselves, I have long ago given elsewhere, and have not Room to repeat them here.

S C H O L I U M.

275. Although the Business of reverting Serieses is not a Branch of the Doctrine of Fluxions, but, more properly, belongs to common *Algebra*; yet, as it is often useful where Fluxions are concerned, and falls under the general Rules illustrated in the foregoing Pages, I shall here add an Example or two on that Head.

Let, then, $ax+bx^2+cx^3+dx^4+ex^5 \text{ \&c. } = y$; to revert the Series, or, to find x in an Infinite Series expressed in the Powers of y .

Here, by writing y^n for x , the Indices of the Powers of y , in the Equation, will be $n, 2n, 3n, \text{ \&c. }$ and 1; therefore $n=1$. and the Differences are 0, 1, 2, 3, 4, 5, $\text{ \&c. }$ and so the Series to be assumed, in this Case, is $Ay+By^2+Cy^3+Dy^4 \text{ \&c. }$ Which being involved and substituted for the respective Powers of x (neglecting, every where, all such Powers of x and y as exceed the highest you would have the Series carry'd to) there arises

$$\begin{array}{rcl}
 aAy + aBy^2 + aCy^3 + aDy^4 & \text{ \&c. } & \\
 * \quad + bA^2y^2 + 2bABy^3 + 2bACy^4 & \text{ \&c. } & \\
 & + bB^2y^4 & \text{ \&c. } \\
 * \quad * \quad + cA^3y^3 + 3cA^2By^4 & \text{ \&c. } & \\
 * \quad * \quad * \quad + dA^4y^4 & \text{ \&c. } &
 \end{array} \left. \vphantom{\begin{array}{rcl} aAy + aBy^2 + aCy^3 + aDy^4 \\ * \quad + bA^2y^2 + 2bABy^3 + 2bACy^4 \\ & + bB^2y^4 \\ * \quad * \quad + cA^3y^3 + 3cA^2By^4 \\ * \quad * \quad * \quad + dA^4y^4 \end{array}} \right\} = y$$

Whence,

Whence, by comparing the homologous Terms, $A =$

$$\frac{1}{a}; B = -\frac{b}{a^3}; C (= -\frac{2bAB + cA^3}{a}) = \frac{2bb - ac}{a^5};$$

$$D (= -\frac{2bAC + bB^2 + 3cA^2B + dA^4}{a}) = \frac{5abc - 5b^3 - a^2d}{a^7},$$

$$\text{&c. and consequently } x = \frac{y}{a} - \frac{by^2}{a^3} + \frac{2bb - ac}{a^5} \times y^3 \\ - \frac{5b^3 - 5abc + a^2d}{a^7} \times y^4 \text{ &c.}$$

For an Instance of the Use of this Conclusion, let $x = \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \text{ &c.} = y$: Then, a being, in this

Case, $= 1$, $b = -\frac{1}{2}$, $c = \frac{1}{3}$, $d = -\frac{1}{4}$, &c. we shall, by substituting these Values, have $x = y + \frac{y^2}{2} + \frac{y^3}{6} +$

$\frac{y^4}{24} \text{ &c.}$ From whence, when y is given, x will also be

given; provided the Value of y be sufficiently small *. * Art. 92.

Example 2. Let there be given $ax + by + cx^2 + dxy + ey^2 + fx^3 + gx^2y + hxy^2 + iy^3 + kx^4 + lx^3y \text{ &c.} = 0$; to find y .

By assuming $y = Ax + Bx^2 + Cx^3 + Dx^4 \text{ &c.}$ and proceeding as above, A will be found $= -\frac{a}{b}$, $B = -$

$$\frac{c + dA + eA^2}{b}, C = -\frac{dB + 2eAB + f + gA + bA^2 + iA^3}{b}, D =$$

$$-\frac{dC + 2eAC + eB^2 + gB + 2hAB + 3iA^2B + k + lA +$$

$$\frac{mA^2 + nA^3 + pA^4}{b}, \text{ &c.}$$

Example

Example 3. Lastly, let $x^m + bx^{m+p} + cx^{m+2p} + dx^{m+3p} + \text{&c.} = z$.

Here, in order to determine the Form of the Series to be assumed, let z^n be wrote for x in the given Equation, according to the usual Method; and then the Exponents, supposing z transposed, will be $1, nm, nm + np, nm + 2np, nm + 3np, \text{&c.}$ respectively; whereof the two least (1 and nm) being made equal to each other, n is found $= \frac{1}{m}$; and the Differences are $\frac{p}{m}, \frac{2p}{m},$

$\frac{3p}{m}, \text{&c.}$ Whence the Series to be assumed for x is

$$x^{\frac{1}{m}} + Bx^{\frac{1+p}{m}} + Cx^{\frac{1+2p}{m}} + Dx^{\frac{1+3p}{m}} + \text{&c.} \quad (\text{for it is evident, by Inspection, that the Coefficient (A) of the first Term must here be an Unit.})$$

This Series being therefore raised to the several Powers of x , in the given Equation, by *Art.* 108. and the Coefficients of the homologous Terms in the new Equation compared together;

it will be found that, $B = -\frac{b}{m}, C = \frac{1 + m + 2p \times bb - 2mc}{2m^2},$

$$D = -\frac{2m^2 + 9mp + 9p^2 + 3m + 6p + 1 \times b^3}{6m^3} +$$

$$\frac{1 + m + 3p \times bc}{m^2} - \frac{d}{m}, \text{&c.}$$

From the general Value of x , found above, innumerable Theorems, for reverting particular Forms of Serieses, may be deduced.

Thus, if $x + bx^2 + cx^3 + dx^4, \text{&c.} = z$; then (m being $= 1$ and $n = 1$) x is $= z - bz^2 + \frac{2bb - c}{2} \times z^3 - \frac{5b^3 - 5bc + d}{6} \times z^4 \text{&c.}$

And

And, if $x + bx^3 + cx^5 + dx^7 + \text{etc.} = z$; (m being $=1$, and $p=2$) $x = z - bz^3 + \frac{3bb}{3} - \frac{cxz^5}{12b^3 - 8cb + d} \times z^7 \text{ etc.}$

Also, if $x^{\frac{1}{2}} + bx^{\frac{3}{2}} + cx^{\frac{5}{2}} + dx^{\frac{7}{2}} \text{ etc.} = z$; then (m being $=\frac{1}{2}$ and $p=1$) $x = z^2 - 2bz^4 + \frac{7bb - 2c}{30b^3 - 18bc + 2d} \times z^8 \text{ etc. etc.}$

276. It may be observed that, in all these Forms of Serieses, the first Term is without a Coefficient (which renders the Conclusion much more simple.) Therefore, when the Series to be reverted has a Co-efficient in its first Term, the whole Equation must be first of all divided thereby: Thus, if the Equation was $3x - 6x^2 + 8x^3 - 13x^4 \text{ etc.} = y$; by dividing the whole by 3 it will become $x - 2x^2 + \frac{8x^3}{3} - \frac{13x^4}{3} \text{ etc.} = \frac{1}{3}y$.

Where, putting $z = \frac{1}{3}y$, we have, by Form. I. $x = z + 2z^2 + \frac{16}{3}z^3 \text{ etc.} = \frac{y}{3} + \frac{2y^2}{9} + \frac{16y^3}{81} \text{ etc.}$

SECTION III.

Of the Comparison of Fluents, or the Manner of finding one Fluent from another.

277. **W**E have, already, pointed out the most remarkable Forms of Fluxions whose Fluents are explicable in finite Terms^{*}; and also ^{• Art. 77.} shewn the Use of Infinite Serieses in approximating the ^{78. 83. 84.} Values of such Fluents as do not come under any of ^{and 85.} those Forms[†]: But this last Method (as is before ^{† Art. 99.} hinted) being troublesome, and attended with many Obstacles; Mathematicians have therefore invented, and shewn, the Way of deriving one Fluent from another: Which is of good Advantage when the Fluent

X

sought

fought can be referred to one, like those in *Art.* 126. and 142. expressing the Logarithm of a Number, or the Arch of a Circle; since the Trouble of an infinite Series is, then, avoided.

As the Subject here proposed is of such a Nature, that it would be very tedious and difficult, if not altogether impracticable, to lay down Rules and Precepts for all the various Cases; I shall deliver, what I have to offer thereon, by way of *Problems*; beginning with some very easy ones, for the Sake of the *young Proficient*.

P R O B. I.

278. The Fluent of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ being given (by *Art.* 126.)

'tis proposed to find, from thence, the Fluent of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$.

Let both the Numerator and Denominator of $\frac{x^2\dot{x}}{\sqrt{a^2+x^2}}$, be multiply'd by x , so that the Quantity

without the Vinculum, in the Fluxion, $\frac{x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$,

thus transformed, may become some constant Part of the Fluxion of the highest Term under the Vinculum: Which Part, in this Case, being $\frac{1}{4}$, let $\frac{1}{4}$ of the Fluxion of the first Term under the Vinculum (or $\frac{1}{2}a^2x\dot{x}$) be therefore added to the Numerator, in order to have the

Whole, $\frac{\frac{1}{2}a^2x\dot{x} + x^3\dot{x}}{\sqrt{a^2x^2+x^4}}$, a complete Fluxion; and then the

* *Art.* 77. Fluent thereof, by the common Rule*, will be $\frac{1}{2}$

$\sqrt{a^2x^2+x^4} = \frac{1}{2}x\sqrt{a^2+x^2}$: But, from *this*, we are

now to deduct the Fluent of the Quantity $\frac{\frac{1}{2}a^2x\dot{x}}{\sqrt{a^2x^2+x^4}}$

($= \frac{\frac{1}{2}a^2\dot{x}}{\sqrt{a^2+x^2}}$) that was added: Which Fluent, as

that

that of $\frac{\dot{x}}{\sqrt{a^2+x^2}}$ is given = hyp. Log. $x + \sqrt{a^2+x^2}$ *, * Art. 126.

will be = $\frac{1}{2} a^2 \times$ hyp. Log. $x + \sqrt{a^2+x^2}$; and consequently the Fluent sought = $\frac{1}{2} x \sqrt{a^2+x^2} - \frac{1}{2} a^2 \times$
hyp. Log. $x + \sqrt{a^2+x^2}$. Q. E. I.

PROB. II.

279. Let it be proposed to find the Fluent of $\frac{x^2 \dot{x}}{\sqrt{a^2-x^2}}$,
from that of $\frac{\dot{x}}{\sqrt{a^2-x^2}}$; given by Art. 142.

By proceeding as above, and adding $-\frac{1}{2} a^2 x \dot{x}$ to the Numerator, we have $-\frac{\frac{1}{2} a^2 x \dot{x} - x^3 \dot{x}}{\sqrt{a^2 x^2 - x^4}}$; whereof the Fluent, by the common Rule, is $-\frac{1}{2} \sqrt{a^2 x^2 - x^4}$ ($= -\frac{1}{2} x \sqrt{a^2 - x^2}$): From which deducting the Fluent of $-\frac{\frac{1}{2} a^2 x \dot{x}}{\sqrt{a^2 x^2 - x^4}}$, or $-\frac{\frac{1}{2} a^2 \dot{x}}{\sqrt{a^2 - x^2}}$ (given $= -\frac{1}{2} a^2 \times$ Arc (A) whose Radius is Unity and Sine $= \frac{x}{a}$ *) there comes out $\frac{1}{2} a^2 A - \frac{1}{2} x \sqrt{a^2 - x^2}$. * Art. 142.

Q. E. I.

280. In the same Manner, if the Power without the Vinculum, in the Expression whose Fluent is sought, exceeds that in the other Expression given, by the Exponent under the Vinculum, or by any Multiple of it, the required Fluent may be determined, by one, or by several Operations, according to the Value of the said Multiple.

Thus, if the Fluent of $\frac{x^4 \dot{x}}{\sqrt{a^2-x^2}}$ was sought; then, because the Index of x , without the Vinculum, exceeds
X 2that

that in $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$ by twice the Exponent under the Vinculum, the required Fluent may be had from that of $\frac{\dot{x}}{\sqrt{a^2 - x^2}}$, at two Operations; by the first whereof,

we have already found the Fluent of $\frac{x^2 \dot{x}}{\sqrt{a^2 - x^2}}$ to be =

$\frac{1}{2} a^2 A - \frac{1}{2} x \sqrt{a^2 - x^2}$: Whence, putting this Value = B , and proceeding as before, we also get $-\frac{1}{4} \sqrt{a^2 x^6 - x^8}$

$+ \frac{3}{4} a^2 B = -\frac{1}{4} x^3 \sqrt{a^2 - x^2} - \frac{3a^2 x}{8} \sqrt{a^2 - x^2} +$

$\frac{3a^4 A}{8} = \frac{3a^4 A - 2xx + 3aa \times x \sqrt{a^2 - x^2}}{8} =$ the true

Fluent of $\frac{x^4 \dot{x}}{\sqrt{a^2 - x^2}}$.

PROB. III.

281. Supposing the Fluent of $\overline{a + cz^m}^m \times z^{pn-1} \dot{z}$ to be given = A , to find the Fluent of $\overline{a + cz^m}^m \times z^{pn+n-1} \dot{z} = B$ (where the Exponent of z , without the Vinculum is increased by the Exponent under the Vinculum.)

Let the Part affected by the Vinculum be multiply'd by z^{mq} , and the Part without be divided by the same Quantity; then our Fluxion will be transformed to $\overline{az^q + cz^{n+q}}^m \times z^{pn+n-mq-1} \dot{z} = B$: Where let q be, now, so taken that the Exponent $(n+q)$ of the highest Power of z under the Vinculum may be equal to $(pn+n-mq)$ that of the Power without the Vinculum $+ 1$; that is, let $q = \frac{pn}{m+1}$: Then (by Art. 77. if the first Term

under

under the *Vinculum* was constant, the Fluent of the said Expression, or its Equal $\overline{az^q + cz^{n+q}}^m \times z^{n+q-1} z$,

would be had = $\frac{\overline{az^q + cz^{n+q}}^{m+1}}{m+1 \times n+q \times c}$. But the

Fluxion hereof, supposing both Terms to be variable

(as they actually are) is $\overline{az^q + cz^{n+q}}^m \times z^{n+q-1} z + \frac{qa}{n+q \times c} \times \overline{az^q + cz^{n+q}}^m \times z^{q-1} z$ (by the common

Rule.) Therefore $\frac{\overline{az^q + cz^{n+q}}^{m+1}}{m+1 \times n+q \times c} - \frac{qa}{n+q \times c} \times$

Flu. of $\overline{az^q + cz^{n+q}}^m \times z^{q-1} z = B$; that is,

$\frac{\overline{a+cz^n}^{m+1} \times z^{q+m+q}}{m+1 \times n+q \times c} - \frac{qa}{n+q \times c} \times \text{Flu. } \overline{a+cz^n}^m \times$

$z^{mq+q-1} z = B$; or, by substituting for q ,

$\frac{\overline{a+cz^n}^{m+1} \times z^{pn}}{m+p+1 \times nc} - \frac{pa}{m+p+1 \times c} \times \text{Flu. } \overline{a+cz^n}^m \times$

$z^{pn-1} z = B$: But the Flu. of $\overline{a+cz^n}^m \times z^{pn-1} z$ is

given = A ; therefore, lastly, $\frac{\overline{a+cz^n}^{m+1} \times z^{pn}}{m+p+1 \times nc} -$

$\frac{paA}{m+p+1 \times c} = B$.

Q. E. I.

282. If the Quantity under the *Vinculum* be a Multinomial, $a + cz^n + dz^{2n} + ez^{3n} \&c$. Then, since

the Fluxion of $\overline{a+cz^n + dz^{2n} + ez^{3n} \&c}^{m+1} \times z^{pn}$ is

$\frac{\overline{a+cz^n + dz^{2n} + ez^{3n} \&c}^{m+2} \times z^{pn}}{m+1 \times nc z^{n-1} z + 2ndz^{2n-1} z + 3nez^{3n-1} z \&c. \times$

X 3

$$\begin{aligned} & \overline{a+cz^n + dz^{2n} \mathcal{E}c.}^m \times z^{pn} + \overline{a+cz^n + dz^{2n} \mathcal{E}c.}^{m+1} \\ & \times pnz^{p^{n-1}z} = \\ & \left\{ \begin{array}{l} * \quad m+1 \times nc z^{pn+n-1}z + m+1 \times 2ndz^{pn+2n-1}z \mathcal{E}c. \\ pna z^{p^{n-1}z} + pnc z^{pn+n-1}z + pndz^{pn+2n-1}z \mathcal{E}c. \end{array} \right\} \end{aligned}$$

$\times \overline{a+cz^n + dz^{2n} \mathcal{E}c.}^m$, it is evident, that, if the Fluents of $z^{p^{n-1}z}$, $z^{pn+n-1}z$, $z^{pn+2n-1}z \mathcal{E}c.$ drawn

into the general Multiplicator $\overline{a+cz^n + dz^{2n} \mathcal{E}c.}^m$, be denoted by A , B , C , D , $\mathcal{E}c.$ the Fluent of the Whole-Quantity exhibited above (which Fluent is

$\overline{a+cz^n + dz^{2n} + ez^{3n} \mathcal{E}c.}^{m+1} \times z^{pn}$) will also be expressed by $pnaA + p+m+1 \times ncB + p+2m+2 \times ndC + p+3m+3 \times neD \mathcal{E}c.$ Therefore, if there be given as many of the Fluents A , B , C , $D \mathcal{E}c.$ as there are

Terms in $a+cz^n + dz^{2n} + ez^{3n} \mathcal{E}c.$ minus one, that other Fluent, be it which it will, will also be given from hence. Thus, if $d=0$, $e=0$, $\mathcal{E}c.$ and the Value of

$$\begin{aligned} & A \text{ be given, we shall have } \overline{a+cz^n}^{m+1} \times z^{pn} = pnaA + \\ & \overline{a+cz^n}^{m+1} \times z^{pn} \\ & p+m+1 \times ncB; \text{ and consequently } B = \frac{\overline{a+cz^n}^{m+1} \times z^{pn}}{p+m+1 \times nc} \\ & - \frac{paA}{p+m+1 \times c}, \text{ the very same as before.} \end{aligned}$$

PROB. IV.

283. The Fluent of $\overline{a+cz^n}^m \times z^{p^{n-1}z}$ being given (as in the preceding Problem) to determine, from thence, the Fluent of $\overline{a+cz^n}^m \times z^{pn+vn-1}z$; supposing v to denote a whole positive Number.

Let

Let $\overline{a+cz^n}^{m+1}$ be denoted by M ; also put $p+1 = \dot{p}$, $\dot{p}+1 (p+2) = \ddot{p}$, $\ddot{p}+1 (p+3) = \dddot{p}$ &c. and let the Fluents of $\overline{a+cz^n}^m \times z^{\dot{p}n-1} \dot{z}$, $\overline{a+cz^n}^m \times z^{\ddot{p}n-1} \ddot{z}$, $\overline{a+cz^n}^m \times z^{\ddot{p}n-1} \ddot{z}$, $\overline{a+cz^n}^m \times z^{\ddot{p}n-1} \ddot{z}$, &c. be represented by A, B, C, D , &c. respectively. Then, since

$\frac{Mz^{\dot{p}n}}{m+\dot{p}+1 \times nc} - \frac{\dot{p}aA}{m+\dot{p}+1 \times c} = B$ (by the preceding Prob.) it follows, from the very same Argument, that

$$\frac{Mz^{\dot{p}n}}{m+\dot{p}+1 \times nc} - \frac{\dot{p}aB}{m+\dot{p}+1 \times c} = C$$

$$\frac{Mz^{\ddot{p}n}}{m+\ddot{p}+1 \times nc} - \frac{\ddot{p}aC}{m+\ddot{p}+1 \times c} = D$$

Hence, by writing the Value of B in the second Equation, we have

$$\frac{Mz^{\dot{p}n}}{m+\dot{p}+1 \times nc} - \frac{\dot{p}aMz^{\dot{p}n}}{m+\dot{p}+1 \times m+\dot{p}+1 \times nc} + \frac{\dot{p}\dot{p}a^2A}{m+\dot{p}+1 \times m+\dot{p}+1 \times c^2} = C.$$

In the same Manner, by substituting this Value for C in the 3d Equation, we get

$$\frac{Mz^{\ddot{p}n}}{m+\ddot{p}+1 \times nc} - \frac{\ddot{p}aMz^{\dot{p}n}}{m+\ddot{p}+1 \times m+\dot{p}+1 \times nc^2} +$$

$$\frac{\overline{\overline{ppa^2 Mz^{pn}}}}{m + \overline{p} + 1 \times m + \overline{p} + 1 \times \overline{m + p + 1} \times nc^3} = D.$$

$$\overline{m + p + 1 \times m + \overline{p} + 1 \times m + \overline{p} + 1 \times c^3}$$

Where the Law of Continuation is manifest; and from whence it appears that the Value of any of the Quantities *B*, *C*, *D*, *E*, &c. or the Fluent expressed in a general Manner, will be

$$\frac{Mz^{qn}}{m + q + 1 \times nc} + \frac{qa Mz^{q-1 \times n}}{m + q + 1 \times m + q \times nc} + \frac{q \times q - 1 \times a^2 Mz^{q-2 \times n}}{m + q + 1 \times m + q \times m + q - 1 \times nc^3} \\ (v) \pm \frac{p \times p + 1 \times p + 2 \times p + 3 (v) \times a^v A}{m + p + 1 \times m + p + 2 \times m + p + 3 (v) \times c^v}; \text{ or,} \\ \frac{a + cz^n}{s + 1 \times nc} \times z^{pn} - \frac{qaz^{vn-2n}}{sc} + \frac{q \cdot q - 1 \times a^2 z^{vn-3n}}{s \cdot s - 1 \times c^2} \\ - \frac{q \cdot q - 1 \cdot q - 2 \times a^3 z^{vn-4n}}{s \cdot s - 1 \cdot s - 2 \times c^3} (v) \pm \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2} \\ \frac{p+3}{t+3} (v) \times \frac{a^v A}{c^v} : \text{ Where, } A = \text{Fluent of } \overline{a + cz^n}^m$$

$\times z^{pn-1}$, $q = p + v - 1$, $s = q + m$, $t = p + m + 1$; and where the Sign of the last Term (in which *A* is found) must be taken + or - according as *v* is an even or odd Number: Note, also, that the Parenthesis (*v*) is put to express the Number of Terms, or Factors, to which the Series, or Product, preceding it, is to be continued. The like Notation is to be understood in other Cases of the same Kind, when they hereafter occur.

The

The same otherwise.

284. Let $q = p + v - 1$, and let $\overline{a + cz^n}^{m+1} \times$
 $Rz^n + Sz^{q-n} + Tz^{q-n-2n} \dots + \Delta z^{pn} + \beta A$, be
 assumed for the Fluent sought: Then, by taking the
 Fluxion thereof, you will have $\overline{m+1} \times nc z^{n-1} \times \overline{a + cz^n}^m$
 $\times Rz^n + Sz^{q-n} \dots + \Delta z^{pn} + \overline{a + cz^n}^{m+1} \times$
 $qnzRz^{q-n-1} + qn-n \times Sz^{q-n-1} \dots + pnz\Delta z^{p-n-1}$
 $+ \beta \times \overline{a + cz^n}^m \times z^{p-n-1}$; which must be $= \overline{a + cz^n}^m$
 $\times z^{p+vn-1}$ (or $\overline{a + cz^n}^m \times z^{q+n-1}$) the Fluxion
 proposed: Whence, dividing the whole Equation by
 $\overline{a + cz^n}^m \times z^{n-1}$, and transposing, there comes out

$$\left. \begin{array}{l} \overline{m+1} \times nc \times Rz^n + Sz^{q-n} + Tz^{q-n-2n} \dots + \Delta z^{pn} \\ \overline{a + cz^n} \times qnRz^{q-n-1} + qn-n \times Sz^{q-n-1} \dots + pn\Delta z^{p-n-1} \\ - z^{qn} \quad * \quad * \quad + \beta z^{p-n} \end{array} \right\} \parallel 0$$

Which, reduced, and the homologous Terms united,
 becomes

$$\left. \begin{array}{l} \overline{m+q+1} \times nc R \\ - 1 \end{array} \right\} \times z^{qn} + \left. \begin{array}{l} \overline{m+q} \times nc S \\ + qna R \end{array} \right\} \times z^{q-n} +$$

$$\left. \begin{array}{l} \overline{m+q-1} \times nc T \\ + qn-n \times a S \end{array} \right\} \times z^{q-n-2n} \dots + \left. \begin{array}{l} pna \Delta \\ + \beta \end{array} \right\} \times z^{p-n}$$

$= 0$: Where, by making $\overline{m+q+1} \times nc R - 1 = 0$,

$\overline{m+q} \times nc S + qna R = 0$, &c. we have $R = \frac{1}{\overline{m+q+1} \times nc}$,

$S = -\frac{qaR}{\overline{m+q} \times c}$, $T = -\frac{\overline{q-1} \times aS}{\overline{m+q-1} \times c}$; or (putting

$m+q$

$$m+q=s) R = \frac{1}{s+1 \times nc}, S = -\frac{qaR}{sc} = -\frac{qa}{s+1 \times snc^2},$$

$$T = -\frac{q-1 \times aS}{s-1 \times c} = \frac{q \times q-1 \times a^2}{s+1 \times s \times s-1 \times nc^3}, \text{ \&c.}$$

Where, because the Exponent of the first Term of the Equation is qn ($pn+vn-n$) and that of the last Term (in which Δ and β are concerned) $=pn$, it follows that the Number of Coefficients to be taken as above (whereof Δ is the last) is expressed by v ; From which last, the Value of β is given $= -pna\Delta$.

But, from the Law of the said Coefficients, $R, S, \dots \Delta$, it appears that the Value of Δ (whose Place from the Beginning is denoted by v) will be $= \pm$

$$\frac{q.q-1.q-2 \dots q-v+2}{s+1.s.s-1 \dots s-v+2} \times \frac{a^{n-v}}{nc^n} = \pm$$

$$\frac{q.q-1.q-2 \dots p+1}{s+1.s.s-1 \dots p+m+1} \times \frac{a^{n-v}}{nc^n}: \text{ And therefore } \beta$$

$$(= -pna\Delta) = \pm \frac{q.q-1.q-2 \dots p+1.p}{s+1.s.s-1 \dots p+m+1}$$

$$\times \frac{a^n}{c^n} = \pm \frac{p.p+1.p+2.p+3(v)}{t.t+1.t+2.t+3(v)} \times \frac{a^n}{c^n} \quad (\text{putting}$$

$p+m+1=t$, as before.) Now, if the several Values of $R, S, T, \dots$ and β , thus found, be substituted in the assumed Expression, you will have the very same Conclusion as in the preceding Article.

COROLLARY I.

285. Since q is $=p+v-1$, the Fluent $\overline{a+cz^n}^{m+1} \times Rz^{qn} + Sz^{qn-n} \dots + \Delta z^{pn} + \beta A$, given above, may be expressed by $N \times Rz^{vn-n} + Sz^{vn-2n} + Tz^{vn-3n} (v) + \beta A$; where $N = \overline{a+cz^n}^{m+1} \times z^{pn}$, $R =$

$$\frac{1}{m+p+v \times nc}, S = - \frac{p+v-1.aR}{m+p+v-1.c}, T = -$$

$$\frac{p+v-2.a\delta}{m+p+v-2.c} : \text{And, where the Coefficient } (\beta) \text{ of the}$$

given Fluent (A) will always be expressed by the last of the Quantities $R, S, T \dots \Delta$, multiply'd by $-pna$: This is evident, because it is found that $\beta = -pna\Delta$. And the same thing will also appear from the several particular Cases (*in Art.* 283.) for the Values of B, C and D : In each of which the Coefficient of the last Term (where A is concern'd) is to *that* of the Term immediately preceding it, in the constant Ratio of pa to

$\frac{1}{n}$, or of pna to Unity.

COROLLARY II.

286. If the Value of c be negative, the general Fluent (*in Art.* 283.) when $a+cz^n = 0$ (provided $m+1, n$, and p be positive) will become barely $= \pm \frac{p}{t} \times \frac{p+1}{t+1} \times$

$\frac{p+2}{t+2} (v) \times \frac{a^v A}{c^v}$; because, in this Circumstance, all

the Terms multiply'd by $\overline{a+cz^n}^{m+1}$ intirely vanish. If, therefore, b be wrote for $-c$ (to render the Expression more commodious) we shall have $\frac{p}{t} \times \frac{p+1}{t+1}$

$\times \frac{p+2}{t+2} (v) \times \frac{a^v A}{b^v}$ for the true Fluent of $\overline{a-bz^n}^m \times$

$z^{pn+vn-1}z$, generated while bz^n , from Nothing, becomes $= a$: Where A denotes the Fluent of $\overline{a-bz^n}^m \times z^{pn-1}z$, generated in the same time; and where

$t =$

$t = p + m + 1$. Hence it follows that the Fluent of $\frac{a - bz^n}{a - bz^n} \times z^{pn-1} \dot{z} \times e + fz^n + gz^{2n} + bz^{3n} \&c.$ (where e, f, g are any given Quantities) will be $= A \times$
 $e + \frac{paf}{tb} + \frac{p.p+1.a^2g}{t.t+1.b^2} + \frac{p.p+1.p+2.a^3h}{t.t+1.t+2.b^3} + \&c.$ in the
 foremention'd Circumstance.

PROB. V.

287. The Fluent (A) of $\frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z}$ being given, to find the Fluent of $\frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z}$; supposing r to denote a whole positive Number.

Since $\frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z} = \frac{a + cz^n}{a + cz^n} \times \frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z}$, it is evident that $\frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z} = \frac{a + cz^n}{a + cz^n} \times az^{pn-1} \dot{z} + \frac{a + cz^n}{a + cz^n} \times cz^{pn+n-1} \dot{z}$: Whose Fluent (by Prob. 3.)

$$\text{is } aA + \frac{\frac{a + cz^n}{a + cz^n} \times z^{pn}}{m+p+1 \times n} - \frac{paA}{m+p+1} =$$

$$\frac{\frac{a + cz^n}{a + cz^n} \times z^{pn}}{p+m+1 \times n} + \frac{m+1 \times aA}{p+m+1}. \text{ In like Manner, if}$$

this Fluent, of $\frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z}$, be denoted by B , that of $\frac{a + cz^n}{a + cz^n} \times z^{pn-1} \dot{z}$ by C , &c. it will appear that

$$\frac{\frac{a + cz^n}{a + cz^n} \times z^{pn}}{p+m+2 \times n} + \frac{m+2 \times aB}{p+m+2} = C;$$

$$\frac{\frac{a + cz^n}{a + cz^n} \times z^{pn}}{p+m+3 \times n} + \frac{m+3 \times aC}{p+m+3} = D, \&c. \text{ Whence,}$$

by substituting these Values, one by one, as in the preceding

ceding Problem, and putting $\mathcal{Q} = a + cz^n$, we get

$$C = \frac{\mathcal{Q}^{m+2} z^{pn}}{p+m+2.n} + \frac{m+2 \times a \mathcal{Q}^{m+1} z^{pn}}{p+m+2.p+m+1.n} + \frac{m+2.m+1 \times a^2 A}{p+m+2.p+m+1} ; D = \frac{\mathcal{Q}^{m+3} z^{pn}}{p+m+3.n} + \frac{m+3 \times a \mathcal{Q}^{m+2} z^{pn}}{p+m+3.p+m+2.n} + \frac{m+3.m+2 \times a^2 \mathcal{Q}^{m+1} z^{pn}}{p+m+3.p+m+2.p+m+1.n} + \frac{m+3.m+2.m+1 \times a^3 A}{p+m+3.p+m+2.p+m+1} , \&c. \text{ Whence it is}$$

evident, by Inspection, that the Fluent of $\overline{a+cz^n}^{m+r} \times z^{pn-1} z$, expressed in a general Manner, will be $\frac{\mathcal{Q}^{m+r} z^{pn}}{p+m+r.n} + \frac{m+r \times a \mathcal{Q}^{m+r-1} z^{pn}}{p+m+r \times p+m+r-1.n} \&c.$ Which,

by putting $m+r=f$, $p+m+r=g$, and making $\mathcal{Q}^{m+1} \times z^{pn}$ a general Multiplier, will be reduced to $\mathcal{Q}^{m+1} \times z^{pn} \times \frac{\mathcal{Q}^{f-1}}{g^n} + \frac{f \times a \mathcal{Q}^{f-2}}{g.g-1.n} + \frac{f.f-1 \times a^2 \mathcal{Q}^{f-3}}{g.g-1.g-2.n} (r) + \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \times \frac{m+3}{p+m+3} (r) a^r A$; where it appears (from the foregoing Values of B, C, and D) that the Coefficient of A is always equal to the last Term of the preceding Series, multiply'd by $\overline{m+1} \times na$ (instead of $\mathcal{Q}^{m+1} z^{pn}$.) $\mathcal{Q} \text{ E. I.}$

COROLLARY.

288. If c be negative, so that $\mathcal{Q}$, or its Equal, $a + cz^n$, may become equal to Nothing, the Fluent will,
2 in

in that Circumstance, be barely $= \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2}$

$\times \frac{m+3}{p+m+3} (r) \times a^r A$; provided the Values of $m+1$, p , and n are positive: Or, if c , p and n be positive, and $m+r+p$ negative, the same Expression will exhibit the true Value of the whole Fluent, generated while z , from Nothing, becomes infinite.

PROB. VI.

289. The same being given as in the preceding Problems;

'tis propos'd to find the Fluent of $a+cz^n$ $\times$ $z^{p-m-1} z$.

If $-r$ be wrote instead of r , in the last Article, we shall have $m-r=f$, $p+m-r=g$, and $\mathcal{Q}^{m+1} z^p \times \frac{\mathcal{Q}^{-r-1}}{g^n} + \frac{f \times a \mathcal{Q}^{-r-2}}{g \cdot g-1 \cdot n} (-r) + \frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} (-r) \times a^{-r} A$, expressing the required Fluent in this Case.

But $\frac{m+1}{p+m+1} \times \frac{m+2}{p+m+2} \&c.$ continued to $-r$

Factors, signifies the same thing as the Product continued downwards, or the contrary way, to r Factors, according to the same Law: And therefore is $= \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2} (r)$. After the same

Manner we have $\frac{\mathcal{Q}^{-r-1}}{g^n} + \frac{f \times a \mathcal{Q}^{-r-2}}{g \cdot g-1 \cdot n} (-r) =$
 $-\frac{\mathcal{Q}^{-r}}{f+1 \cdot na} - \frac{g+1 \cdot \mathcal{Q}^{-r+1}}{f+1 \cdot f+2 \cdot na^2} - \frac{g+1 \cdot g+2 \cdot \mathcal{Q}^{-r+2}}{f+1 \cdot f+2 \cdot f+3 \cdot na^3} (r)$

$$(r) \text{ and consequently the Fluent itself} = \frac{\mathcal{Q}^{m+1} z^{pn} \times}{\frac{-\mathcal{Q}^{-r}}{f+1.na} - \frac{g+1.\mathcal{Q}^{1-r}}{f+1.f+2.na^2} - \frac{g+1.g+2.\mathcal{Q}^{2-r}}{f+1.f+2.f+3.na^3}} (r) \\ + \frac{p+m}{m} \times \frac{p+m-1}{m-1} \times \frac{p+m-2}{m-2} (r) \times \frac{A}{a^r} \mathcal{Q}.E.I.$$

COROLLARY.

290. It appears from hence that the Coefficient of A , the given Fluent, will always be equal to *that* of the last Term of the preceding Series, multiply'd by $p+m \times n$: For, seeing the Coefficient of the said last Term (whose Distance from the first, inclusive, is denoted by r) must be

$$\frac{g+1.g+2.g+3 \dots g+r-1}{f+1.f+2.f+3 \dots f+r} \times \frac{1}{na^r} \text{ (by the Law of}$$

the Series) where $f+r=m$ and $g+r-1=p+m-1$ (as appears from above) it follows, by inverting the Order

of both Progressions, that $\frac{p+m-1.p+m-2 \dots (r-1)}{m.m-1.m-2 \dots (r)}$

$\times \frac{1}{na^r}$ will also express the same Coefficient: Which,

multiply'd by $p+m \times n$, gives $\frac{p+m.p+m-1.p+m-2 \dots (r)}{m.m-1.m-2 \dots (r)}$

$\frac{1}{a^r}$, the very Coefficient of A , above determined. The

Use of this Conclusion will be seen in what follows.

PRO-

PROB. VII.

291. *The same being, still, given; to find the Fluent of*

$$\overline{a+cz^n}^m \times z^{pn-vn-1}z.$$

By proceeding as in the last Problem, the required Fluent of $\overline{a+cz^n}^m \times z^{pn-vn-1}$ is derived from *that of* $\overline{a+cz^n}^m \times z^{pn+vn-1}z$ (given by Prob. 4.) and comes out

$$\begin{aligned} &= \mathcal{Q}^{m+1} z^{pn} \times \frac{z^{-vn}}{q+1.na} - \frac{s+2.cz^{n-vn}}{q+1.q+2.na^2} + \\ &\frac{s+2.s+3.c^2z^{2n-vn}}{q+1.q+2.q+3.na^3} (v) \pm \frac{t-1}{p-1} \times \frac{t-2}{p-2} \times \frac{t-3}{p-3} (v) \\ &\times \frac{c^v A}{a^v} : \text{Where, } \mathcal{Q} = a+cz^n, q=p-v-1, s=m+q, \\ &t=p+m+1 : \text{And where, the Coefficient of } A \text{ is equal} \end{aligned}$$

to that of the last of the preceding Terms, multiply'd by $-m+p \times nc$. If the Manner of deducing the required Fluent, in this, and the last, Problem, should not appear sufficiently plain and satisfactory to the Beginner; the same Conclusions may be, otherwise, brought out; by finding A , in Terms of B , C , or D , from the several particular Equations in *Art. 283.* or, by assuming a descending Series, instead of an ascending one. *Vid. Art. 284.*

PROB. VIII.

292. *The same being, still, given; to find the Fluent of*

$$\overline{a+cz^n}^{m+r} \times z^{pn+vn-1}z.$$

Let the Fluent of $\overline{a+cz^n}^m \times z^{pn+vn-1}z$ (given by *Prob. 4.*) be denoted by B , and that required by F :
Then,

Then, if $p+v$ be put $=\dot{p}$, the Value of F (the Fluent of $\overline{a+cz^n}^{m+r} \times z^{\dot{p}n-1} \dot{z}$) will be given from that of B (the Fluent of $\overline{a+cz^n}^m \times z^{\dot{p}n-1} \dot{z}$) by writing B for A and $\dot{p}$ for p , in *Art.* 287. Whence we get $F=\mathcal{Q}^{m+1}$

$$\begin{aligned} & z^{\dot{p}n} \times \frac{\mathcal{Q}^{r-1}}{g^n} + \frac{fa\mathcal{Q}^{r-2}}{g.g-1.n} + \frac{f.f-1.a^2\mathcal{Q}^{r-3}}{g.g-1.g-2.n} (r) + \\ & \frac{m+1}{\dot{p}+m+1} \times \frac{m+2}{\dot{p}+m+2} \times \frac{m+3}{\dot{p}+m+3} (r) \times a^r B: \text{Where} \\ & \dot{p}=p+v, f=m+r, g(=\dot{p}+m+r)=p+m+v+r, \text{ and } \mathcal{Q}=a+cz^n. \end{aligned}$$

Which Fluent, by substituting the Value of B (in *Prob.* 4.) becomes $F=\mathcal{Q}^{m+1} z^{\dot{p}n} \times \frac{\mathcal{Q}^{r-1}}{g^n} + \frac{fa\mathcal{Q}^{r-2}}{g.g-1.n}$
 $+ \frac{f.f-1.a^2\mathcal{Q}^{r-3}}{g.g-1.g-2.n} (r) + \frac{m+1}{\dot{p}+m+1} \times \frac{m+2}{\dot{p}+m+2} (r)$

$$\begin{aligned} & \times a^r \times \mathcal{Q}^{m+1} z^{\dot{p}n} \times \frac{z^{vn-n}}{s+1.nc} - \frac{qaz^{vn-2n}}{s+1.snc^2} + \frac{q.q-1.a^2z^{vn-3n}}{s+1.s.s-1.nc^3} \\ & (v) \pm \frac{m+1}{\dot{p}+m+1} \times \frac{m+2}{\dot{p}+m+2} (r) \times a^r \times \frac{\dot{p}}{t} \times \frac{\dot{p}+1}{t+1} \end{aligned}$$

$(v) \times \frac{a^v A}{c^v}$: Where $q=p+v-1, s=m+q=m+p+v-1,$

and $t=p+m+1$; and where the Sign of the last Term is $+$ or $-$ according as v is an even or odd Number.
 $\mathcal{Q}. E. I.$

292. If the last Term of the first Series, exclusive of the general Multiplier $\mathcal{Q}^{m+1} z^{\dot{p}n}$, be denoted by β , the Multiplier, $\frac{m+1}{\dot{p}+m+1} \times \frac{m+2}{\dot{p}+m+2} (r) \times a^r$, to

* Art. 287. the second Series will be $= \frac{m+1}{s+1} \times na\beta^*$; and therefore the first Term of this Series, including its Multiplier, is $= \frac{m+1 \cdot a\beta \mathcal{Q}^{m+1} z^{\dot{p}n+v}}$: Which, if R

be put to denote the last Term $\beta \mathcal{Q}^{m+1} z^{\dot{p}n+v}$ of the first Series (with its Multiplier) will be expounded by $\frac{m+1 \cdot aR}{s+1 \cdot cz^n}$. Hence it follows, that the Fluent of

$$\frac{a + cz^n}{s+1} \times z^{\dot{p}n+v-1} z, \text{ given above, will also be truly}$$

$$\text{expressed by } \frac{\mathcal{Q}^{m+r} \times z^{\dot{p}n+v}}{gn} + \frac{f}{g-1} \times \frac{aH}{\mathcal{Q}} + \frac{f-1}{g-2} \times$$

$$\frac{aI}{\mathcal{Q}} + \frac{f-2}{g-3} \times \frac{aK}{\mathcal{Q}} (r) + \frac{m+1}{s+1} \times \frac{aR}{cz^n} - \frac{q}{s} \times$$

$$\frac{aS}{cz^n} - \frac{q-1}{s-1} \times \frac{aT}{cz^n} - \frac{q-2}{s-2} \times \frac{aV}{cz^n} (v) \pm$$

$$\frac{m+1 \cdot m+2 \cdot m+3 (r) \times \dot{p} \cdot \dot{p}+1 \cdot \dot{p}+2 (v)}{\dot{p}+m+1 \cdot \dot{p}+m+2 (r) \times t \cdot t+1 \cdot t+2 (v)} \times \frac{a^{v+r} A}{c^v} :$$

Where $H, I, K, L, \dots, R, S, T, V, \&c.$ represent the Terms immediately preceding those where they stand, under their proper Signs: R being the last Term of the first Series; also $f = m+r, g = m+r+p+v, q = p+v-1, s = m+q, t = m+p+1$, and $\mathcal{Q} = a + cz^n$.

COROLLARY II.

293. Since the Divisor, $\overline{p+m+1} \cdot \overline{p+m+2} (r) \times \overline{t.t+1.t+2} (v)$, of the last Term of the Fluent (by substituting for t and $\dot{p}$ &c.) is $= \overline{p+m+1} \cdot \overline{p+m+2} (v) \times \overline{p+v+m+1} \cdot \overline{p+v+m+2} (r)$: Where, the last Factor ($\overline{p+m+v}$) of the first Progression, is less by Unity than the first Factor of the Second; it is evident that the said second Progression is only a Continuation of the first to r more Factors: And so, the last Term of the Fluent, where A is found, is truly expressed by $\pm \frac{\overline{p.p+1.p+2} (v) \times \overline{m+1.m+2.m+3} (r)}{\overline{m+p+1.m+p+2.m+p+3} (v+r)} \times \frac{a^{v+r} A}{c^v}$.

Hence it follows, that the Fluent of $\overline{a+cz^n}^{m+r} \times z^{pn+vn-1} \dot{z}$, or that of $\overline{a-bz^n}^{m+r} \times z^{pn+vn-1} \dot{z}$ (making $c=-b$) will, when $a-bz^n$ becomes equal to Nothing, be barely =

$$\frac{\overline{p.p+1.p+2} (v) \times \overline{m+1.m+2.m+3} (r)}{\overline{m+p+1.m+p+2.m+p+3} (v+r)} \times \frac{a^{v+r} A}{b^v} :$$

A being the Fluent of $\overline{a-bz^n}^m \times z^{pn-1} \dot{z}$, in that Circumstance, v and r whole positive Numbers, and p and $m+1$ any positive Numbers, either whole or broken.

SCHOLIUM.

294. If the Fluent of $\overline{a+cz^n}^{m+r} \times z^{pn-1} \dot{z}$ (given by *Prob. 5.*) be denoted by C ; then (F) the Fluent of

$\overline{a+cz^n}^m \times z^{pn+vn-1} \dot{z}$ (where $m'=m+r$) will be had, from C (by *Prob. 4.*) according to a new Form, different

ferent from those already given. And, by following the same Method, the Fluents of $\overline{a+cz^n}^{m-r} \times z^{pn+vn-1}$, $\overline{a+cz^n}^{m+r} \times z^{pn-vn-1}$, and $\overline{a+cz^n}^{m-r} \times z^{pn-vn-1}$ may also be found, each, according to two different Forms, from a Combination of the corresponding Cases in the foregoing Problems.

But, as it is extremely tiresome to repeat the same thing, again and again, where such a Number of Symbols are necessarily concerned, I shall here put down one Solution to each Case (because of their Use) leaving the Process and the other Forms (which contain no new Difficulty) to *Those* who will be at the Trouble to set about them.

$$\begin{aligned}
 1^\circ. \text{ The Fluent of } \overline{a+cz^n}^{m-r} \times z^{pn+vn-1} \text{ is} = \\
 - \frac{\overline{a+cz^n}^{m-r+1} \times z^{pn+vn}}{f+1.na} + \frac{g+1}{f+2} \times \frac{\mathcal{Q}H}{a} + \frac{g+2}{f+3} \times \frac{\mathcal{Q}I}{a} (r) \\
 - \frac{\mathcal{Q}R}{cz^n} - \frac{q}{s} \times \frac{aS}{cz^n} - \frac{q-1}{s-1} \times \frac{aT}{cz^n} - \frac{q-2}{s-2} \times \frac{aV}{cz^n} (v) \\
 + \frac{s+1}{m} \times \frac{s}{m-1} \times \frac{s-1}{m-2} (r) \times \frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{a^v A}{\overline{a+cz^n}^v}.
 \end{aligned}$$

Where $H, I, K, L, \dots, R, S, T, \&c.$ denote the Terms immediately preceding those where they stand, under their proper Signs; R being the last Term of the first Series, also $\mathcal{Q} = \overline{a+cz^n}$, $f = m-r$, $g = p+m+v-r$, $q = p+v-1$, $s = m+p+v-1$, $t = p+m+1$, and $A =$ the given Fluent of $\overline{a+cz^n}^m \times z^{pn-1}$.

2°. The

$$2^{\circ}. \text{ The Fluent of } \overline{a+cz^n}^{m+r} \times z^{pn-vn-1} z \text{ is } =$$

$$\frac{\mathcal{Q}^{m+r+1} \times z^{pn-vn}}{q+1.na} - \frac{s+2}{q+2} \times \frac{Hcz^n}{a} - \frac{s+3}{q+3} \times \frac{Icz^n}{a} (v)$$

$$- \frac{Rcz^n}{\mathcal{Q}} + \frac{f}{g-1} \times \frac{a\mathcal{S}}{\mathcal{Q}} + \frac{f-1}{g-2} \times \frac{aT}{\mathcal{Q}} + \frac{f-2}{g-3} \times \frac{aV}{\mathcal{Q}} (r)$$

$$+ \frac{s+2}{q+1} \times \frac{s+3}{q+2} \times \frac{s+4}{q+3} (v) \times \frac{m+1}{t} \times \frac{m+2}{t+1} \times \frac{m+3}{t+2} (r) \times \frac{\overline{-c}^v A}{a^{v-r}}.$$

Where $q=p-v-1$, $s=m+r+q$, $f=m+r$, $g=p+m+r$, and the rest as in the preceding Case.

$$3^{\circ}. \text{ The Fluent of } \overline{a+cz^n}^{m-r} \times z^{pn-vn-1} z \text{ is } =$$

$$- \frac{\mathcal{Q}^{m-r+1} \times z^{pn-vn}}{f+1.na} + \frac{g+1}{f+2} \times \frac{\mathcal{Q}H}{a} + \frac{g+2}{f+3} \times \frac{\mathcal{Q}I}{a} (r)$$

$$- \frac{s+1}{q+1} \times \frac{\mathcal{Q}R}{a} - \frac{s+2}{q+2} \times \frac{\mathcal{S}cz^n}{a} - \frac{s+3}{q+3} \times \frac{\mathcal{T}cz^n}{a} (v)$$

$$+ \frac{\overline{t-1.t-2.t-3.t-4.t-5} (r+v)}{m.m-1.m-2 (r) \times p-1 p-2.p-3 (v)} \times \frac{\overline{-c}^v A}{a^{r+v}}.$$

In which $f=m-r$, $g=m+p-r-v$, $q=p-v-1$, $s=q+m$, and the rest as before.

295. From what has been delivered in this Section, the Fluents of various Forms of Fluxions may be exhibited, by means of circular Arcs and Logarithms.

For, since the Fluents of $\overline{a+cz^n}^{-1} \times z^{\frac{1}{2}n-1} z$, $\overline{a+cz^n}^{-\frac{1}{2}} \times z^{\frac{1}{2}n-1} z$, and $\overline{a+cz^n}^{-\frac{1}{2}} \times z^{-1} z$ (which I call original Ones) are all of them explicable by one, or the other, of these two Kinds of Quantities (as will appear farther on) those of $\overline{a+cz^n}^{-1 \pm r} \times z^{\frac{1}{2}n \pm vn-1} z$, $\overline{a+cz^n}^{-\frac{1}{2} \pm r} \times z^{\frac{1}{2}n \pm vn-1} z$, and $\overline{a+cz^n}^{-\frac{1}{2} \pm r} \times z^{\pm vn-1} z$ will also be given from thence, by the fore-

Y 3

going

going Theorems. Whence the most useful Forms of Fluents in *Cotes's Harmonia Mensurarum* will be obtained, besides some others, more general than any, of the same Kind, put down by that sagacious Author.

Here follow a few Examples of some of the most useful Cases.

E X A M P L E I.

296. Let the Fluxion given be $\frac{z^{2v} \dot{z}}{\sqrt{d^2 \pm z^2}}$ (or $\overline{d^2 \pm z^2}^{-\frac{1}{2}} \times z^{2v} \dot{z}$) v being any whole positive Number.

Then, the Fluent of $\overline{d^2 \pm z^2}^{-\frac{1}{2}} \times \dot{z}$, or $\frac{\dot{z}}{\sqrt{d^2 \pm z^2}}$

being = hyp. Log. $\frac{z + \sqrt{d^2 \pm z^2}}{d}$; or, equal to the

* Art. 126. Arch whose Sine is $\frac{z}{d}$ and Radius Unity*; according 142.

as the second Term, in $d^2 \pm z^2$, is positive or negative; let A be, therefore, taken to denote the said Arch, or

Logarithm; and let $\overline{d^2 \pm z^2}^{-\frac{1}{2}} \times \dot{z}$ be compared with

$\overline{a + cz^n}^m \times z^{pn-1} \dot{z}$ (whose Fluent is, all along, supposed to be given = A) and you will have $a = d^2$, $c = \pm 1$, $n = 2$, $m = -\frac{1}{2}$, $2p - 1 = 0$, and therefore $p = \frac{1}{2}$: Whence, by substituting those Values in Art. 283. we

likewise get $q(p + v - 1) = \frac{2v - 1}{2}$, $s(m + q) = v$

-1 , $t(m + p + 1) = 1$; and, consequently, the Fluent

sought = $\overline{d^2 \pm z^2}^{\frac{1}{2}} \times \pm \frac{z^{2v-1}}{2v} - \frac{2v-1 \cdot d^2 z^{2v-3}}{2v \cdot 2v-2} \pm$

$\frac{2v-1 \cdot 2v-3 \cdot d^4 z^{2v-5}}{2v \cdot 2v-2 \cdot 2v-4} - \frac{2v-1 \cdot 2v-3 \cdot 2v-5 \cdot d^6 z^{2v-7}}{2v \cdot 2v-2 \cdot 2v-4 \cdot 2v-6}$

(v)

(v) $\pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} (v) \times d^{2v}$: In which the last Term is negative, when the given Fluxion is $\frac{z^{2v} \dot{z}}{\sqrt{d^2 + z^2}}$, and v , at the same time, an odd Number; but in all other Cases, affirmative.

EXAMPLE II.

297. Let $z^{2v} \dot{z} \sqrt{d^2 \pm z^2}$ (or $\overline{d^2 \pm z^2}^{-\frac{1}{2}+1} \times z^{2v} \dot{z}$) be propounded.

Here, denoting the Fluent of $\overline{d^2 \pm z^2}^{-\frac{1}{2}} \dot{z}$ by A (as above) and comparing $\overline{d^2 \pm z^2}^{-\frac{1}{2}+1} \times z^{2v} \dot{z}$, with $\overline{d^2 \pm z^2}^{m+r} \times z^{pn+qn-1} \dot{z}$ (Vid. Prob. 8.) we have $r=1$, and the rest as in the last Example: Whence also $\dot{p} (p+v) = v + \frac{1}{2}$, $f(m+r) = \frac{1}{2}$, $g=v+1$, $Q=d^2 \pm z^2$, and the Fluent itself $= \frac{z^{2v+1} \sqrt{d^2 \pm z^2}}{2v+2} \pm \frac{d^2 R}{2vz^2}$
 $\mp \frac{\overline{2v-1} \cdot d^2 S}{2v-2 \cdot z^2} \mp \frac{\overline{2v-3} \cdot d^2 T}{2v-4 \cdot z^2} (1+v) \pm \frac{1}{2} \times \frac{3}{4}$
 $\times \frac{5}{6} (v) \times \frac{d^{2v+2} A}{2v+2}$ * ($R, S, T, \&c.$ being the pre- * Art. 292.

ceding Terms with their Signs) $= \frac{\sqrt{d^2 \pm z^2}}{2v+2} \times z^{2v+1} \pm$

$$\frac{d^2 z^{2v-1}}{2v} - \frac{2v-1 \cdot d^4 z^{2v-3}}{2v \cdot 2v-2} + \frac{2v-1 \cdot 2v-3 \cdot d^6 z^{2v-5}}{2v \cdot 2v-2 \cdot 2v-4}$$

$(v+1) \pm \frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8} (v) \times \frac{d^{2v+2} A}{2v+2}$: Where

the Sign of the last Term must be regulated as in the

preceding Example—If the Fluent of $\frac{z^{-vn}z}{\sqrt{d^2 \pm z^2}}$, or of $z^{-vn}z \sqrt{d^2 \pm z^2}$ (in which the Exponent is negative) be required; the Answer will be had in finite Terms, independent of A , by Art. 85.

EXAMPLE III.

298. Wherein the Fluxion proposed is $\overline{d^n - z^n}^{-\frac{1}{2} + r} \times z^{\frac{1}{2}n + vn - 1}z$; r and v being any whole positive Numbers.

Since the Fluent of $\overline{d^n - z^n}^{-\frac{1}{2}} \times z^{\frac{1}{2}n - 1}z$ (as will appear hereafter) is truly expressed by $\frac{2}{n} \times \text{Arch}$, whose

Sine is $\frac{z^{\frac{1}{2}n}}{d^{\frac{1}{2}n}}$ and Radius Unity, let this Value be de-

noted by A ; and then, by writing d^n for a , -1 for c , $-\frac{1}{2}$ for m , and $\frac{1}{2}$ for p , in Art. 292. we shall have f

$$(m+r) = \frac{2r-1}{2}, g(m+p+r+v) = r+v, q(p+v-1)$$

$$= \frac{2v-1}{2}, s(m+q) = v-1, t(p+m+1) = 1, \mathcal{Q}$$

$(a+cz^n) = d^n - z^n$, and the Fluent, itself, equal to

$$\frac{\mathcal{Q}^{r-\frac{1}{2}} z^{vn+\frac{1}{2}n}}{r+v.n} + \frac{\overline{2r-1}}{r+v-1} \times \frac{\frac{1}{2}d^n H}{\mathcal{Q}} + \frac{2r-3}{r+v-2} \times$$

$$\frac{\frac{1}{2}d^n I}{\mathcal{Q}} + \frac{2r-5}{r+v-3} \times \frac{\frac{1}{2}d^n K}{\mathcal{Q}} (r) - \frac{\frac{1}{2}d^n R}{vz^n} + \frac{2v-1}{v-1} \times$$

$$\frac{\frac{1}{2}d^n S}{z^n} + \frac{2v-3}{v-2} \times \frac{\frac{1}{2}d^n T}{z^n} (v) + \frac{1.3.5.7(r) \times 1.3.5.7(v)}{2.4.6.8.10.12(r+v)}$$

* Art. 293. * $\times d^{rn+vn}A$: In which $H, I, K, \dots R, S, T, \&c.$ denote the preceding Terms with their Signs; K being the

the last Term of the first Series. Hence, because all the Terms, but the last, vanish, when $Q=0$, it follows that

the whole Fluent of $\overline{a^n - z^n}^{r-\frac{1}{2}} \times z^{vn+\frac{1}{2}n-1} z$, generated while z , from Nothing, becomes equal to d , is truly expressed by $\frac{1.3.5.7 (r) \times 1.3.5.7 (v)}{2.4.6.8.10.12 (r+v)} \times d^{rn+vn} A$, or

by $\frac{1.3.5.7 (r) \times 1.3.5.7 (v)}{2.4.6.8.10.12 (r+v)} \times \frac{d^{rn+vn} G}{n}$; G being the Semi-Periphery of the Circle whose Radius is Unity.

EXAMPLE IV.

299. Let it be required to find the whole Fluent of

$$\frac{\overline{a - bz^n}^m \times z^{pn-1} z}{\overline{d + kz^n}^\beta}, \text{ generated while } bz^n, \text{ from No-}$$

thing, becomes $=a$; that of $\overline{a - bz^n}^m \times z^{pn-1} z$ being given ($=A$.)

Here, by expanding $\overline{d + kz^n}^\beta$, our given Fluxion becomes $= \overline{a - bz^n}^m \times z^{pn-1} z$ into $d^{-\beta} \times 1 - \frac{\beta kz^n}{d} + \frac{\beta. \beta + 1. k^2 z^{2n}}{1. 2. d^2} - \frac{\beta. \beta + 1. \beta + 2. k^3 z^{3n}}{1. 2. 3. d^3} \&c.$

Which Series being compared with $e + fz^n + gz^{2n} \&c.$

(Vid. Art, 286.) we have $e=1$, $f=-\frac{\beta k}{d}$, $g=$

$\frac{\beta. \beta + 1. k^2}{1. 2. d^2}$, $\&c.$ and consequently the Fluent sought

(by substituting these Values) equal to $\frac{A}{a^\beta}$ into $1 -$

$$\frac{p}{t} \times \frac{\beta}{1} \times \frac{ak}{bd} + \frac{p}{t} \cdot \frac{p+1}{t+1} \times \frac{\beta}{1} \cdot \frac{\beta+1}{2} \times \left(\frac{ak}{bd} \right)^2 -$$

$$\frac{p}{1} \cdot \frac{p+1}{1+1} \cdot \frac{p+2}{1+2} \times \frac{\beta}{1} \cdot \frac{\beta+1}{2} \cdot \frac{\beta+2}{3} \times \left(\frac{ak}{bd} \right)^3 + \text{Ec.} (\text{e}$$

being = $p+m+1$.)

Here the Values of $m+1$, n and p are supposed po-

• Art. 286. sitive *; and it is requisite that $1 + \frac{ak}{bd}$ should also be

positive; otherwise the Fluent will fail. Although the Series brought out above runs on to Infinity, yet it may be sum'd, in many Cases: Thus, if the given Fluxion

be $\frac{a-bz^n}{d+kz^n} \times z^{\frac{1}{2}n-1}$; then, the foresaid Series be-

coming $1 - \frac{1}{2} \times \frac{ak}{bd} + \frac{1}{2} \times \frac{1}{4} \times \left(\frac{ak}{bd} \right)^2 - \text{Ec.}$ its Sum

will be $1 + \frac{ak}{bd}$: And consequently $\frac{A}{d} \times 1 + \frac{ak}{bd}$

= the Fluent sought: Where, A (the whole Fluent of $\frac{a-bz^n}{d+kz^n} \times z^{\frac{1}{2}n-1}$) being = $\frac{1}{n\sqrt{b}} \times$ Semi-Peri-

phery of the Circle whose Radius is Unity, the Fluent given above will, therefore, be = $\frac{1}{n\sqrt{bd^2+adk}}$

$\times$ by the same Semi-Periphery. If the Reader is desirous to see a further Application of the Summation of Serieses, to the finding of Fluents, I must refer him to my *Dissertations* (where it is handled in a general Manner) having neither Room nor Inclination to treat of it here.

SECTION IV.

Of the Transformation of Fluxions.

301. **B**Y the Transformation of Fluxions may be understood, the reducing *any* fluxional Quantity to a different, or more commodious, Form; according to which Sense, a great Part of the second Section would properly fall under this Head. But, *what* is here proposed, and *what* is commonly meant by the Transformation of Fluxions, *is*, the Method of ordering those Kinds of Expressions which involve one variable Quantity *only* with its Fluxion; which, yet, are so affected by radical Signs, that the Fluent, without an Infinite Series, would be impracticable, were it not for a new Substitution, or some other Kind of Transformation, whereby the given Fluxion is render'd more manageable.

Something of this Sort has been already touch'd upon in *Art. 83*. And in what follows I shall farther point out and exemplify the principal Cases wherein such a *Procedure* will be of Service.

302. *If the Number of Dimensions of the variable Quantity, without the Vinculum, increased by Unity, be some aliquot Part, or Parts, of the Dimensions of the same Quantity, under the Vinculum, the Fluxion will be reduced to a better Form by substituting for that Power of the variable Quantity, which arises by dividing its Exponent, under the Vinculum, by the Denominator of the Fraction expressing the said aliquot Part, or Parts.*

Thus, if the Fluxion propounded be $\frac{z^{\frac{1}{2}n-1} \dot{z}}{\sqrt{c^n \pm z^n}}$; by

substituting $x = z^{\frac{1}{2}n}$, and taking the Fluxion of both Sides of the Equation, we have $\dot{x} = \frac{1}{2}nz^{\frac{1}{2}n-1} \dot{z}$; and therefore $z^{\frac{1}{2}n-1} \dot{z} = \frac{\dot{x}}{\frac{1}{2}n}$: Which Value, with that of z^n , being wrote for their Equals, in the given Fluxion, it will

will be transformed to $\frac{\dot{x}}{\frac{1}{2}n\sqrt{c^n \pm x^2}}$: Which, putting $a=c^{\frac{1}{2}n}$ (to make the Terms homologous), is also expressed by $\frac{\dot{x}}{\frac{1}{2}n\sqrt{a^2 \pm x^2}}$: Whereof the Fluent will be given by *Art.* 126. or *Art.* 142. according as the Sign of x^2 is positive or negative.

303. If the Power of the variable Quantity under the *Vinculum* has a Coefficient, it will be best to bring that Coefficient without the *Vinculum*.

Ex. 2. Where let the Fluxion given be $\frac{z^{\frac{1}{2}n-1}\dot{z}}{\sqrt{a+cz^n}}$:

Which, by bringing c without the *Vinculum*, becomes

$\frac{z^{\frac{1}{2}n-1}\dot{z}}{c^{\frac{1}{2}}\sqrt{\frac{a}{c}+z^n}}$: From whence, by putting $x=z^{\frac{1}{2}n}$

and proceeding as above, we get $\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}}\sqrt{\frac{a}{c}+x^2}}$:

Whose Fluent, by *Art.* 126. is $\frac{1}{\frac{1}{2}nc^{\frac{1}{2}}} \times \text{hyp. Log. } x +$

$\sqrt{\frac{a}{c}+x^2}$. This, by restoring z , becomes $\frac{2}{nc^{\frac{1}{2}}} \times$

$\text{hyp. Log. } z^{\frac{1}{2}n} + \sqrt{\frac{a}{c}+z^n}$. Which, corrected) by

supposing it = 0 when $z=0$) gives, at length, $\frac{2}{nc^{\frac{1}{2}}} \times$

$\text{hyp. Log. } z^{\frac{1}{2}n} + \sqrt{\frac{a}{c}+z^n} - \text{hyp. Log. } \sqrt{\frac{a}{c}} =$

$\frac{2}{nc^{\frac{1}{2}}} \times \text{hyp. Log. } \sqrt{\frac{cz^n}{a}} + \sqrt{1+\frac{cz^n}{a}}$ for the true

Fluent of the Quantity proposed.

But,

But, when c is a negative Quantity, this Fluent fails, because the square Root of c is to be extracted. In

this Case $\frac{\dot{x}}{\frac{1}{2}nc \sqrt{\frac{a}{c} + x^2}}$ must be transformed to

$\frac{\dot{x}}{\frac{1}{2}n\sqrt{-c} \times \sqrt{\frac{a}{-c} - x^2}}$: And then its Fluent (by

Art. 142.) will be had = $\frac{1}{\frac{1}{2}n\sqrt{-c}} \times$ the Arch of a

Circle whose Radius is Unity, and Right-Sine =

$$\frac{x}{\sqrt{\frac{a}{-c}}} = \sqrt{\frac{-cx^n}{a}}$$

Ex. 3. Let the given Fluxion be $\frac{\dot{z}}{z\sqrt{a+cz^n}}$.

Which, by bringing c without the Vinculum, and put-

ting $x = z^{\frac{1}{n}}$, is transformed to $\frac{\dot{x}}{\frac{1}{2}nc x \sqrt{\frac{a}{c} + x^2}}$:

Whereof the Fluent, by Art. 126. is $\frac{1}{n\sqrt{a}} \times \text{hyp. Log.}$

$$\frac{\sqrt{\frac{a}{c}} - \sqrt{\frac{a}{c} + x^2}}{\sqrt{\frac{a}{c}} + \sqrt{\frac{a}{c} + x^2}} = \frac{1}{n\sqrt{a}} \times \text{hyp. Log.}$$

$$\frac{\sqrt{a} - \sqrt{a+cz^n}}{\sqrt{a} + \sqrt{a+cz^n}}.$$

But here, when c is positive, the Numerator will be negative; in which Case it will be proper to change its Signs, and exprefs the Fluent by

$$\frac{1}{n\sqrt{a}} \times \text{hyp. Log.} \frac{\sqrt{a+cz^n} - \sqrt{a}}{\sqrt{a+cz^n} + \sqrt{a}}.$$

That, such

an

an Alteration of the Signs can make no Difference in the Fluxion, is evident from the Nature of Logarithms;

because the Fluxion of the *Log.* of $-x$ ($= \frac{-\dot{x}}{-x} = \frac{\dot{x}}{x}$)

is the same with *that* of the *hyp. Log.* of x . It will be proper to observe farther, that, instead of the Logarithm above derived, any one of the following, equal, Quantities may be taken; viz. *hyp. Log.* $\frac{\sqrt{a+cz^n} - \sqrt{a}}{cz^n}$

(found by multiplying both the Numerator and Denominator of the forefaid Logarithm by $\sqrt{a+cz^n} - \sqrt{a}$)

$= 2 \times \text{hyp. Log.} \frac{\sqrt{a+cz^n} - \sqrt{a}}{\sqrt{cz^n}}$ (by the Nature

of Logarithms) $= 2 \times \text{hyp. Log.} \frac{\sqrt{cz^n}}{\sqrt{a+cz^n} + \sqrt{a}}$

(by multiplying, equally, by $\sqrt{a+cz^n} + \sqrt{a}$)

But, take which of these Forms you will, the Fluent fails when a is negative; because the general Multiplier

$\frac{1}{n\sqrt{a}}$ is then impossible. In this Case the Fluent of

$\frac{\dot{x}}{\frac{1}{2}nc^{\frac{1}{2}} \times x \sqrt{\frac{a}{c} + x^2}}$, or its Equal $\frac{\dot{x}}{z\sqrt{a+cz^n}}$, will

be given by *Art.* 142. and is expounded by $-\frac{1}{\frac{1}{2}nc^{\frac{1}{2}} \sqrt{\frac{-a}{c}}}$

$\times A = \frac{2}{n\sqrt{-a}}$; where A denotes the *Arch* whose

Radius is Unity, and Secant $\frac{x}{\sqrt{\frac{-a}{c}}} (= \sqrt{\frac{cz^n}{-a}})$.

In

In the same Manner the Fluent of $\frac{z^{\frac{1}{n}-1}\dot{z}}{a+cz^n}$, is found
 $= \frac{1}{n\sqrt[n]{ac}} \times \text{Arch}$, whose Radius is Unity and Tan-
 gent $\sqrt[n]{\frac{cz^n}{a}}$, or equal to $\frac{1}{n\sqrt[n]{-ca}} \times \text{hyp. Log.}$
 $\frac{\sqrt{a+\sqrt{-cz^n}}}{\sqrt{a-\sqrt{-cz^n}}}$, according as the Value of c is affir-
 mative or negative; a being supposed affirmative.

304. When the Power, or Powers, of the variable Quantity without the Vinculum, or radical Sign, fall, mostly, in the Denominator, it may be of Use to substitute for the Reciprocal of the said Quantity, or for the Quotient which arises by dividing some known Quantity, either, by it, or by some Compound of it in the Denominator.

Ex. 1. Let the proposed Fluxion be $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}}$;
 then, putting $x = \frac{a^2}{z}$, we have $z = \frac{a^2}{x}$, and $\dot{z} = -\frac{a^2\dot{x}}{x^2}$; and consequently $\frac{a^3\dot{z}}{z^2\sqrt{a^2+z^2}} = \frac{-x\dot{x}}{\sqrt{x^2+a^2}}$;
 Whereof the Fluent is $-\sqrt{x^2+a^2} = -\sqrt{\frac{a^4}{z^2}+a^2}$.

Ex. 2. Let the given Fluxion be $\frac{z\dot{z}}{(a+z)^3 \times \sqrt{a^2+az+z^2}}$.

Here, putting $x = \frac{a^2}{a+z}$, we have $z = \frac{aa-ax}{x} =$
 $a \times \frac{a-x}{x}$, $\dot{z} = -\frac{a^2\dot{x}}{x^2}$, $z\dot{z} = -\frac{a^3\dot{x} \times a-x}{x^3}$,
 $\sqrt{a^2+az+z^2} = \frac{a}{x} \sqrt{a^2-ax+x^2}$; and therefore the
 Quantity

Quantity proposed is transformed to $\frac{x^2 \dot{x} - ax \dot{x}}{a^4 \sqrt{a^2 - ax + x^2}}$;

Whose Fluent may be found from a Table of Logarithms; as will appear farther on.

305. If the Fluxion given is affected by two different Surds, and the rational Factor, or the Quantity without the Vinculum, be in a constant Ratio to the Fluxion of the Quantity under the Vinculum of either Surd, or be related to it as in Art. 83. the given Fluxion will be reduced to a more simple Form, by substituting for that Surd.

Ex. 1. Let $\frac{zx \sqrt{b^2 + z^2}}{\sqrt{c^2 - z^2}}$ be propounded.

Then, putting $x = \sqrt{b^2 + z^2}$, we have $z^2 = x^2 - b^2$, $zx = x\dot{x}$, and $\sqrt{c^2 - z^2} = \sqrt{c^2 + b^2 - x^2} = \sqrt{a^2 - x^2}$

(by making $a = \sqrt{c^2 + b^2}$) Whence $\frac{zx \sqrt{b^2 + z^2}}{\sqrt{c^2 - z^2}} =$

• Art. 279. $\frac{x^2 \dot{x}}{\sqrt{a^2 - x^2}}$ *.

Or, if x be put $= \sqrt{c^2 - z^2}$ (instead of $\sqrt{b^2 + z^2}$); then $z^2 = c^2 - x^2$, $zx = -x\dot{x}$, $\sqrt{b^2 + z^2} = \sqrt{b^2 + c^2 - x^2} = \sqrt{a^2 - x^2}$; and consequently $\frac{zx \sqrt{b^2 + z^2}}{\sqrt{c^2 - z^2}} = -\dot{x} \sqrt{a^2 - x^2}$: Whose Fluent is given by Art. 297. or 131.

Ex. 2. Let the given Fluxion be $\overline{a + cz^n}^m \times \overline{e + fz^n}^r \times z^{pn-1} \dot{z}$; supposing p to denote any whole positive Number *.

• Art 83.

In this Case, let that of the two Quantities, $a + cz^n$ and $e + fz^n$, whose Index (m or r) is the most complex (which we will suppose the latter) be put $= x$;

then we shall have $z^n = \frac{x-e}{f}$; $z^{pn-1} \dot{z} = \frac{\dot{x}}{nf}$;

$$z^{p-1} \dot{z} (= z^{p-1} \times z^{n-1} \dot{z}) = \frac{\overline{x-d}^{p-1}}{f^{p-1}} \times \frac{\dot{z}}{nf};$$

$$a + cz^n = a + \frac{cx - ce}{f} = d + \frac{cx}{f} \text{ (by putting } d = a -$$

$$\frac{ce}{f}) \text{ and consequently } d + \frac{cx}{f} \Big)^m \times \frac{\overline{x-e}^{p-1}}{nf^p} \times x^r \dot{z}$$

= the Fluxion proposed: Where, $p-1$ being a whole

positive Number, the Value of $\overline{x-e}^{p-1}$ will therefore be expressed in finite Terms; whence, if m be also a whole positive Number, the Fluent itself will be had in finite Terms: But, if m and r be the Halves of odd Numbers, then the Fluent will be found (from Art. 298 or 294.) by means of circular Arcs and Logarithms.

306. If the given Expression be affected by two Surds wherein the Powers of the variable Quantity are the same, and the rational Quantity, without the Vinculums, be related to the Fluxion of either Surd, as in Art. 83. it may be of Use to substitute for the Quotient, or Ratio, of the two Quantities under the radical Signs; especially, if the Sum of the said radical Signs, or Exponents (supposing both Surds to be reduced to the Denominator) is a whole Number.

Ex. 1. Let the given Fluxion be $\frac{z^2 \dot{z}}{\overline{b^3+z^3}^{\frac{2}{3}} \times \overline{c^3-z^3}^{\frac{4}{3}}}$.

Then, writing $x = \frac{b^3+z^3}{c^3-z^3}$, we have $z^3 = \frac{c^3x-b^3}{1+x}$;

$$3z^2 \dot{z} = \frac{\overline{c^3+b^3} \times \dot{x}}{\overline{1+x}^2}; \overline{b^3+z^3}^{\frac{2}{3}} \times \overline{c^3-z^3}^{\frac{4}{3}} \left(= \overline{b^3+z^3}^{\frac{2}{3}} \right.$$

$$\left. \times \overline{c^3-z^3}^2 = x^{\frac{2}{3}} \times c^3 - \frac{\overline{c^3x-b^3}^2}{\overline{1+x}} \right) = \frac{\overline{b^3+c^3}^2 \times x^{\frac{2}{3}}}{\overline{1+x}^2};$$

Z

and

and consequently $\frac{z^2 \dot{z}}{\sqrt[3]{b^3+z^3} \times \sqrt[3]{c^3-z^3}} = \frac{x^{-\frac{2}{3}} \dot{x}}{3 \times b^3 + c^3}$

Whose Fluent is $\frac{x^{\frac{1}{3}}}{b^3+c^3} = \frac{1}{b^3+c^3} \times 3 \sqrt[3]{\frac{b^3+z^3}{c^3-z^3}}$

Ex. 2. Let there be given $\frac{z^{p-1} \dot{z}}{a+cz^n} \times \frac{e+fz^n}{e+fz^n}^r$

Here, putting $x = \frac{e+fz^n}{a+cz^n}$, you will have $z^n =$

$$\frac{ax-e}{f-cx}; \quad nz^{n-1} \dot{z} = \frac{af-ce \times \dot{x}}{f-cx}^2; \quad z^{p-1} \dot{z} (=z^{p-n-n})$$

$$\times z^{n-1} \dot{z}) = \frac{ax-e}{f-cx}^{p-1} \times \frac{af-ce \times \dot{x}}{n \times f-cx}^2; \quad a+cz^n)^m$$

$$\times e+fz^n)^r (=a+cz^n)^{m+r} \times \frac{e+fz^n}{a+cz^n}^r = a+cx \frac{ax-e}{f-cx}^{r+m}$$

$$\times x^r) = \frac{af-ce}{f-cx}^{m+r} \times x^r; \quad \text{and consequently the}$$

$$\text{Fluxion given} = \frac{ax-e^{p-1} \times f-cx^{m+r-p-1} \times x^{-r} \dot{x}}{n \times af-ce^{m+r-1}}$$

Where, if $m+r$ be a whole positive Number, greater than p (also a whole positive Number) the Fluent will be truly had in finite Terms; because both the Serieses

for the Values of $ax-e^{p-1}$ and $f-cx^{m+r-p-1}$ do in that Case terminate *. But, if r and $m+r-p-1$ be the Halves of whole Numbers, positive or negative, then the Fluent will be given by the last Section.

* Art. 99.

307. A Trinomial is reduced to a Binomial by taking away its middle Term; that is, by substituting for the Sum or Difference of the Power of the variable Quantity

in that Term and half its Coefficient; according as the Signs of the two Terms, where the said Quantity is found, are like, or unlike.

Ex. 1. Let the given Fluxion be $\frac{\dot{z}}{\sqrt{b^2 + cz + z^2}}$; then, putting $x = z + \frac{1}{2}c$, or $z = x - \frac{1}{2}c$, we have $\dot{z} = \dot{x}$, and $\sqrt{b^2 + cz + z^2} (= \sqrt{b^2 + cx - \frac{1}{2}c^2 + x^2 - cx + \frac{1}{4}c^2}) = \sqrt{b^2 - \frac{1}{4}c^2 + x^2}$; whence (making $a^2 = b^2 - \frac{1}{4}c^2$) there results $\frac{\dot{z}}{\sqrt{b^2 + cz + z^2}} = \frac{\dot{x}}{\sqrt{a^2 + x^2}}$: Whose Fluent is given, by Art. 126.

Ex. 2. Let the Fluxion given be $\frac{fz^{n-1}\dot{z}}{\sqrt{a + bz^n + cz^{2n}}}$.

First, by bringing c without the Vinculum, according to Art. 303. we have $\sqrt{a + bz^n + cz^{2n}} = \sqrt{c} \times$

$\sqrt{\frac{a}{c} + \frac{bz^n}{c} + z^{2n}}$: And, by putting $x = z^n + \frac{b}{2c}$, or $z^n = x - \frac{b}{2c}$, we also get $z^{n-1}\dot{z} = \frac{\dot{x}}{n}$, and

$\sqrt{\frac{a}{c} + \frac{bz^n}{c} + z^{2n}} (= \sqrt{\frac{a}{c} + \frac{bx}{c} - \frac{bb}{2cc} + x^2 - \frac{bx}{c} + \frac{bb}{4cc}}) = \sqrt{\frac{a}{c} - \frac{bb}{4cc} + x^2}$: Therefore the

Fluxion, transformed, is $\frac{f\dot{x}}{n\sqrt{c} \times \sqrt{\frac{a}{c} - \frac{bb}{4cc} + x^2}}$:

Whose Fluent is given by Art. 126. when c is a positive Quantity: But, when c is negative, the Fluxion must be

expressed thus, $\frac{f\dot{x}}{n\sqrt{-c} \times \sqrt{\frac{a}{-c} + \frac{bb}{4cc} - x^2}}$:

Answering to Form 2. Art. 142.

Z 2

Ex.

Ex. 3. Let $\frac{fz^{n-1}\dot{z} + gz^{2n-1}\dot{z} + bz^{3n-1}\dot{z} + kz^{4n-1}\dot{z}}{a + bz^n + cz^{2n}}$
be proposed.

Then, following the Steps of the last Example,

$\overline{a + bz^n + cz^{2n}}^m (= c^m \times \frac{a}{c} + \frac{bz^n}{c} + z^{2n})^m$ will

be transformed to $c^m \times \frac{a}{c} - \frac{bb}{4cc} + x^2$: More-

over, z^n being $= x - \frac{b}{2c} = x - d$ (by putting $d =$

$\frac{b}{2c}$) and $z^{n-1}\dot{z} = \frac{\dot{x}}{n}$, we also have $z^{2n-1}\dot{z}$ ($=$

$z^n \times z^{n-1}\dot{z} = \overline{x-d} \times \frac{\dot{x}}{n} = \frac{x\dot{x} - d\dot{x}}{n}$; $z^{3n-1}\dot{z}$

($= z^{2n} \times z^{n-1}\dot{z}$) $= \overline{x-d}^2 \times \frac{\dot{x}}{n} = \frac{x^2\dot{x} - 2dx\dot{x} + d^2\dot{x}}{n}$;

&c. &c. From whence, by substituting these several Values in the given Fluxion, and putting

$\frac{a}{c} - \frac{bb}{4cc} = e^2$, there comes out

$$\frac{f\dot{x} + g \times \overline{x\dot{x} - d\dot{x}} + b \times \overline{x^2\dot{x} - 2dx\dot{x} + d^2\dot{x}} + \mathcal{E}c}{nc^m \times \overline{ee + xx}^m} :$$

Whose Fluent, when the Exponent m is the Half of any Integer, positive or negative, will be found, by means of circular Arcs and Logarithms, from Art. 295.

308. When the Denominator is a rational Trinomial, or Multinomial (that is, when it is without a Vinculum) the best Way of proceeding, for the general Part, is, to resolve the given Fraction into binomial Ones. In order to this, let its Denominator be feigned $= 0$; by means

means of which Equation, whose Roots must be found, you will, by subtracting each Root from the indeterminate Quantity (x), have the binomial Denominators of the required Fractions into which the given One may be resolved: Whose corresponding Numerators, let be denoted $A\dot{x}$, $B\dot{x}$, $C\dot{x}$ &c. then, by putting the Sum of the Fractions, thus arising, equal to the given Fraction, and reducing the whole Equation to the same Denominator, the assumed Quantities A , B , C &c. by comparing the homologous Terms, will be determined.

Ex. 1. Let the given Fraction be $\frac{\dot{x}}{x^2+ax+b}$; then, feigning $x^2+ax+b=0$, the two Roots of the Equation will be $-\frac{1}{2}a - \sqrt{\frac{1}{4}a^2-b}$, and $-\frac{1}{2}a + \sqrt{\frac{1}{4}a^2-b}$: Which being denoted by p and q , we have $x-p$ and $x-q$ for the two binomial Factors whereby x^2+ax+b may be resolved, or by whose Multiplication $(x-p \times x-q)$ the said Quantity is produced.

Let therefore $\frac{A\dot{x}}{x-p} + \frac{B\dot{x}}{x-q}$ be now assumed ($= \frac{\dot{x}}{x^2+ax+b}$) $= \frac{\dot{x}}{x-p \times x-q}$; then, by reducing the whole Equation to one Denomination &c. we get $\overline{A+B \times x\dot{x} - qA + pB + 1 \times \dot{x} = 0}$: Whence A is found $= \frac{1}{p-q}$, $B = \frac{1}{q-p}$; and, consequently,

$$\frac{\dot{x}}{p-q \times x-p} + \frac{\dot{x}}{q-p \times x-q} = \frac{\dot{x}}{x^2+ax+b}.$$

Ex. 2. Let the Quantity proposed be $\frac{x^2\dot{x}}{x^3+ax^2+bx+c}$.

Here, if the binomial Factors whereby x^3+ax^2+bx+c is produced be represented by $x-p$, $x-q$, and $x-r$, and there be assumed $\frac{A\dot{x}}{x-p} + \frac{B\dot{x}}{x-q} + \frac{C\dot{x}}{x-r}$

$$\left(= \frac{x^2 \dot{x}}{x^3 + ax^2 + bx + c} \right) = \frac{x^2 \dot{x}}{x - p \times x - q \times x - r}; \text{ then, in}$$

this Case, we shall have $A \times x - q \times x - r + B \times x - p \times x - r + C \times x - p \times x - q - x^2 = 0$; that is, by Reduction,

$$\left. \begin{array}{l} A \\ B \\ C \\ -1 \end{array} \right\} \times x^2 - \left. \begin{array}{l} q+r \times A \\ p+r \times B \\ p+q \times C \end{array} \right\} \times x + \left. \begin{array}{l} qrA \\ prB \\ pqC \end{array} \right\} = 0.$$

Whence $A+B+C=1$, $A \times q + r + B \times p + r + C \times p + q = 0$, and $Aqr + Bpr + Cpq = 0$. Now, from the first of these Equations, multiply'd by $p+q$, subtract the second, and you will have $A \times p - r + B \times q - r = p+q$: Also, from the first, multiply'd by pq , subtract the third;

then $A \times pq - rq + B \times pq - pr = pq$: Lastly, from the former of the two Equations thus arising, multiply'd by p , subtract the latter, then $A \times pp - pr - pq + qr = pp$,

that is, $A \times p - q \times p - r = p^2$; and consequently $A = \frac{p^2}{p - q \times p - r}$: Whence, by the very same Argument,

$$B = \frac{q^2}{q - p \times q - r}, \text{ and } C = \frac{r^2}{r - p \times r - q}.$$

309. After the same Manner you may proceed in other Cases: But there is an Artifice, or Compendium, for more readily determining the assumed Quantities A, B, C &c. by which a great deal of Trouble is avoided: And that is, by considering the Equation in such Circumstances of the indeterminate Quantity x , when it becomes most simple, or when most of its Terms vanish.

Thus, in the preceding Example, because $A \times x - q \times x - r + B \times x - p \times x - r + C \times x - p \times x - q - x^2 = 0$ (in all Circumstances of x whatever) let x be taken $= p$; then, all the Terms vanishing, except the first and last,

we

have $A \times \overline{p-q} \times \overline{p-r} - p^2 = 0$; and consequently $A = \frac{p^2}{\overline{p-q} \times \overline{p-r}}$; the very same as before.

More universally, let the given Fraction be

$$\frac{x^m \dot{x}}{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ \&c.}} =$$

$$\frac{x^m \dot{x}}{x-p \times x-q \times x-r \times x-s \text{ \&c.}} \quad (\text{where } m \text{ and } n \text{ may}$$

represent any whole positive Numbers whatever, provided the latter be greater than the former.) Then,

$$\text{assuming } \frac{A\dot{x}}{x-p} + \frac{B\dot{x}}{x-q} + \frac{C\dot{x}}{x-r} + \frac{D\dot{x}}{x-s} \text{ \&c.} =$$

$$\frac{x^m \dot{x}}{x^n + ax^{n-1} + bx^{n-2} \text{ \&c.}} \text{ \&c. we shall have } A \times$$

$$x-q \times x-r \times x-s \text{ \&c.} + B \times x-p \times x-r \times x-s \text{ \&c.}$$

$$+ C \times x-p \times x-q \times x-s \text{ \&c.} \text{ \&c.} - x^m = 0: \text{ From}$$

$$\text{whence, by expounding } x \text{ by } p, q, r \text{ \&c. successively,}$$

$$\text{we obtain } A = \frac{p^m}{\overline{p-q} \cdot \overline{p-r} \cdot \overline{p-s} \text{ \&c.}}, \quad B = \frac{q^m}{\overline{q-p} \cdot \overline{q-r} \cdot \overline{q-s} \text{ \&c.}},$$

$C = \frac{r^m}{\overline{r-p} \cdot \overline{r-q} \cdot \overline{r-s} \text{ \&c.}}$ \&c. Whence the Fractions themselves, whereof these Quantities are the Coefficient, or Numerators, will likewise be given.

But the Numerators thus found may, sometimes, be more commodiously expressed by Help of the given Coefficients a, b, c, d \&c. so as to involve only one of the Roots p, q, r \&c. in each Fraction. For, since

$$\overline{x-p} \times \overline{x-q} \times \overline{x-r} \text{ \&c. is supposed, universally, } = x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \text{ \&c. if both Sides of the}$$

Equation be divided by $x-p$, we shall have $\overline{x-q} \times \overline{x-r} \times \overline{x-s} \mathcal{E}c. = \frac{x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} \mathcal{E}c.}{x-p}$

Which last Expression, when x is $= p$, that is, when both the Numerator and the Denominator become equal to Nothing, will, manifestly, be equal to $(\overline{p-q} \times \overline{p-r} \times \overline{p-s} \mathcal{E}c.)$ the Divisor of A . Therefore, if the Fluxion of the Numerator be taken and divided by that of the Denominator, and p be wrote instead of x (*vid.*

Page 155.) we shall have $np^{n-1} + \overline{n-1} \times ap^{n-2} + \overline{n-2} \times bp^{n-3} \mathcal{E}c. = \overline{p-q} \times \overline{p-r} \times \overline{p-s} \mathcal{E}c.$ and there-

fore $A (= \frac{p^m}{\overline{p-q} \cdot \overline{p-r} \cdot \overline{p-s} \mathcal{E}c.}) =$

$\frac{p^m}{np^{n-1} + \overline{n-1} \cdot ap^{n-2} + \overline{n-2} \cdot bp^{n-3} \mathcal{E}c.}$ By the
very same Reasoning $B =$

$\frac{q^m}{nq^{n-1} + \overline{n-1} \cdot aq^{n-2} + \overline{n-2} \cdot bq^{n-3} \mathcal{E}c.}$ $C =$

$\frac{r^m}{nr^{n-1} + \overline{n-1} \cdot ar^{n-2} + \overline{n-2} \cdot br^{n-3} \mathcal{E}c.}$

Hence it appears, that, if the Numerator of the given Fraction be divided by the Fluxion of the Denominator (neglecting $\dot{x}$) and the several Roots p, q, r &c. (found by feigning the Denominator $= 0$) be, successively, substituted in the Quotient, instead of x ; I say, it is evident, that the Quantities so resulting, divided by $x-p, x-q, x-r$ &c. will be the required, binomial, Fractions into which the proposed multinomial One may be resolved.

310. If some of the Roots p, q, r &c. are impossible, which is often the Case, the Fractions thus found, where the impossible Roots are concerned, must
be

be united in Pairs, and so reduced to trinomial Ones, in order to take away the *imaginary Terms*.

Thus, let the Fraction proposed be $\frac{x\dot{x}}{x^3+ax^2+bx+c}$, and let two of the Roots, p and q , of the Equation $x^3+ax^2+bx+c=0$ be impossible: Then, $\frac{A\dot{x}}{x-p} + \frac{B\dot{x}}{x-q} + \frac{C\dot{x}}{x-r}$ being $= \frac{x\dot{x}}{x^3+ax^2+bx+c}$, we shall, by u-

niting the *imaginary Terms*, have $\frac{A+B \times x\dot{x}-Aq+Bp \times x}{x^2-p+q \times x+pq}$

$+ \frac{C\dot{x}}{x-r}$, also, $= \frac{x\dot{x}}{x^3+ax^2+bx+c}$; where the impos-

sible Quantities destroy one another. But, to render this more obvious, let a be taken $= 0$, $b=0$, and $c=$

-1 , so that the given Fraction may become $\frac{x\dot{x}}{x^3-1}$;

then the three Roots (p, q, r) of the Equation, x^3-1

$= 0$, will here be $-\frac{1}{2} + \sqrt{\frac{-3}{4}}$, $-\frac{1}{2} - \sqrt{\frac{-3}{4}}$,

and 1 ; whereof the two former are impossible. More-

over, by dividing the Numerator (x) by the Fluxion of the Denominator ($3x^2$) (according to the *Prescript*) we

have $\frac{1}{3x}$; which, by writing p, q, r successively, in-

stead of x , becomes $\frac{1}{3p}$, $\frac{1}{3q}$ and $\frac{1}{3r}$ for the Values of A ,

B , and C , respectively. Whence $\frac{A+B \times x-dq-Bp}{x^2-p+q \times x+pq}$

$+ \frac{C}{x-r} (= \frac{x}{x^3-1})$ is $= \frac{-\frac{1}{3}x + \frac{1}{3}}{x^2+x+1} + \frac{\frac{1}{3}}{x-1} =$

$\frac{1-x}{3x^2+3x+3} + \frac{1}{3x-3}$. But the same may be, other-

wise

wife, investigated, in a more general Manner; by assuming $\frac{Px+Q}{x^2+x+1} + \frac{R}{x-1} = \frac{x}{x^3-1}$, and proceeding as in the first and second Examples; whence the very same Conclusion will be derived.

If the Fraction proposed be of this Form, *viz.*

$$\frac{z^{pn-1}z}{z^{mn} + az^{mn-n} + bz^{mn-2n}} \text{ \&c.}, \text{ the Method of Resolution will, still, be the same: Since, by putting } x=z^n,$$

the given Expression is reduced to

$$\frac{\frac{1}{n} \times x^{p-1}z}{x^m + ax^{m-1} + bx^{m-2}} \text{ \&c.}$$

It may also be proper to observe, *that*, in very complicated Cases, the Application of two, or more, of the six foregoing Rules, may become necessary. Thus, for Example, if the Fluxion given be

$$\frac{z^{pn-1}z}{a+cz^n} \times \frac{1}{e+fz^n+gz^{2n}}; \text{ by resolving } \frac{1}{e+fz^n+gz^{2n}}$$

into two Binomial Fractions, $\frac{A}{b+z^n} + \frac{B}{k+z^n}$ (*according*

to Art. 308.) we shall have $\frac{z^{pn-1}z}{a+cz^n} \times \frac{1}{e+fz^n+gz^{2n}}$

$$= \frac{Az^{pn-1}z}{a+cz^n} \times \frac{1}{b+z^n} + \frac{Bz^{pn-1}z}{a+cz^n} \times \frac{1}{k+z^n} : \text{Where,}$$

if m be a whole positive Number, greater than p , the Fluent will be had in finite Terms (*by Art. 306. Ex. 2.*)

SECTION V.

The Investigation of Fluents of rational Fractions, of several Dimensions, according to the Forms in Cotes's HARMONIA MENSURARUM.

311. **A**S the Subject here propos'd is a Matter of considerable Difficulty, and has exercis'd the Attention of some of the most celebrated Mathematicians (who, yet, seem to have condescended very little to the Information of their less experienc'd Readers) I shall endeavour to set it in the clearest Light possible: In order to which, it will be requisite to premise the following Lemmas.

LEMMA I.

If the Sine of the Mean of three equi-different Arcs, supposing Radius Unity, be multiply'd by the Double of the Co-sine of the common Difference, and from the Product, the Sine of the lesser Extreme be subtracted, the Remainder will be the Sine of the greater Extreme.

LEMMA II.

312. *If G be taken to denote the greater, and L the lesser, of two unequal Arcs, and their Difference be expressed by D; then will,*

$$1. \frac{\text{Sin. } G. \times \text{Co-f. } D - \text{Sin. } L. \times \text{Rad.}}{\text{Sin. } D} = \text{Co-f. } G$$

$$2. \frac{\text{Co-f. } L \times \text{Rad.} - \text{Co-f. } G \times \text{Co-f. } D}{\text{Sin. } D} = \text{Sin. } G$$

$$3. \frac{\text{Sin. } G. \times \text{Rad.} - \text{Sin. } L \times \text{Co-f. } D}{\text{Sin. } D} = \text{Co-f. } L.$$

The

The former of these two Lemmas may be met with in most Authors upon Trigonometry ; and the latter is nothing more than a Corollary to the *common* Theorems for finding the Sine and Co-sine of the Sum and Difference of two given Arcs ; for which Reasons I shall not stop here to give their Demonstration.

COROLLARY.

313. If any Arch of the Circle, whose Radius is Unity, be denoted by Q , its Sine by s , and its Co-sine by a ; and there be taken $A=2a$, $B=2aA-1$, $C=2aB-A$, $D=2aC-B$, $E=2aD-C$, $F=2aE-D$, &c. it follows (from Lemma 1.) that,

$$\text{Sin. } 2Q (\text{Sin. } Q \times 2a - \text{Sin. } 0) = 2sa - 0 = sA$$

$$\text{Sin. } 3Q (\text{Sin. } 2Q \times 2a - \text{Sin. } Q) = 2sAa - s = sB$$

$$\text{Sin. } 4Q (\text{Sin. } 3Q \times 2a - \text{Sin. } 2Q) = 2sBa - sA = sC$$

$$\text{Sin. } 5Q (\text{Sin. } 4Q \times 2a - \text{Sin. } 3Q) = 2sCa - sB = sD$$

$$\text{Sin. } 6Q (\text{Sin. } 5Q \times 2a - \text{Sin. } 4Q) = 2sDa - sC = sE$$

&c.

&c.

LEMMA III.

314. To resolve the Trinomial $r^{2n} - 2kr^n x^n + x^{2n}$, where n is any whole Number, into simple trinomial Factors.

Since the first Term of the given Quantity $r^{2n} - 2kr^n x^n + x^{2n}$ is divisible, only, by the Powers of r , and the last, only, by those of x ; and it appears that r and x are concerned, exactly, alike ; let therefore $r^2 - 2arx + x^2$ (where r and x are, *also*, alike concerned) be assumed for one of the required trinomial Factors,

whereby $r^{2n} - 2kr^n x^n + x^{2n}$ may be resolved : And let $r^2 - 2arx + x^2 \times r^6 + Ar^7x + Br^6x^2 + Cr^5x^3 + Dr^4x^4 + Cr^3x^5 + Br^2x^6 + Arx^7 + x^8$ (where r and x are, *still*, affected alike) be assumed $= r^{10} - 2kr^5x^5 + x^{10}$ (the Value of n , to render the Operation more perspicuous, being first expressed by 5.)

Then,

by resolving them into more simple ones.

349

Then, by Multiplication and Transposition, we shall have

$$\begin{array}{l}
 r^{10} + Ar^9x + Br^8x^2 + Cr^7x^3 + Dr^6x^4 + Cr^5x^5 + Br^4x^6 + Ar^3x^7 + r^2x^8 \dots * \\
 * - 2Ar^9x - 2AAr^8x^2 - 2ABr^7x^3 - 2ACr^6x^4 - 2ADr^5x^5 - 2ACr^4x^6 - 2ABr^3x^7 - 2AAr^2x^8 - 2Arx^9 \dots * \\
 * + r^8x^2 + Ar^7x^3 + Br^6x^4 + Cr^5x^5 + Dr^4x^6 + Cr^3x^7 + Br^2x^8 + Arx^9 + x^{10} \\
 - r^{10} \dots \dots \dots + 2Ar^5x^5 \dots \dots \dots - x^{10}
 \end{array}
 \left. \vphantom{\begin{array}{l} r^{10} + Ar^9x + Br^8x^2 + Cr^7x^3 + Dr^6x^4 + Cr^5x^5 + Br^4x^6 + Ar^3x^7 + r^2x^8 \dots * \\ * - 2Ar^9x - 2AAr^8x^2 - 2ABr^7x^3 - 2ACr^6x^4 - 2ADr^5x^5 - 2ACr^4x^6 - 2ABr^3x^7 - 2AAr^2x^8 - 2Arx^9 \dots * \\ * + r^8x^2 + Ar^7x^3 + Br^6x^4 + Cr^5x^5 + Dr^4x^6 + Cr^3x^7 + Br^2x^8 + Arx^9 + x^{10} \\ - r^{10} \dots \dots \dots + 2Ar^5x^5 \dots \dots \dots - x^{10} \end{array}} \right\} = 0$$

Whence

by resolving them into more simple ones. 351

∴c. supposing the whole Periphery to be divided into n equal Parts, from the Point F). Hence, if the Co-fines of these several Arcs, expressing all the different Values of a , be represented by b, c and d , ∴c. respectively, we shall have $r^2 - 2brx + x^2, r^2 - 2crx + x^2, r^2 - 2drx + x^2$, ∴c. for the several required Factors, by which

$$r^{2n} - 2kr^n x^n + x^{2n} \text{ may be resolved: and consequently}$$

$$\frac{r^{2n} - 2kr^n x^n + x^{2n}}{r^2 - 2brx + x^2} \times \frac{r^{2n} - 2kr^n x^n + x^{2n}}{r^2 - 2crx + x^2} \times \frac{r^{2n} - 2kr^n x^n + x^{2n}}{r^2 - 2drx + x^2} (n) =$$

$$r^{2n} - 2kr^n x^n + x^{2n}. \quad \text{Q. E. I.}$$

Note, If the Sign of the middle Term $2kr^n x^n$ be positive, the Distance (or Co-sine) ON must be taken on the contrary Side of the Center: But when k is greater than Unity, this Method of Solution fails; since no Co-sine can be greater than the Radius.

COROLLARY I.

315. If $k = 1$, the Arch R (whose Co-sine is k) being $= 0$, the Values of b, c, d , ∴c. will be expressed by the Co-fines of the Arcs $\frac{0}{n}, \frac{P}{n}, \frac{2P}{n}, \frac{3P}{n}$ ∴c. respectively: And our general Equation will here become

$$\frac{r^{2n} - 2r^n x^n + x^{2n}}{r^2 - 2drx + x^2} = \frac{r^{2n} - 2r^n x^n + x^{2n}}{r^2 - 2brx + x^2} \times \frac{r^{2n} - 2r^n x^n + x^{2n}}{r^2 - 2crx + x^2} \times \frac{r^{2n} - 2r^n x^n + x^{2n}}{r^2 - 2drx + x^2} (n).$$

From whence, by extracting the Square-Root, on both Sides, we also have $r^n \propto x^n =$

$$\sqrt[n]{r^2 - 2brx + x^2} \times \sqrt[n]{r^2 - 2crx + x^2} \times \sqrt[n]{r^2 - 2drx + x^2} (n).$$

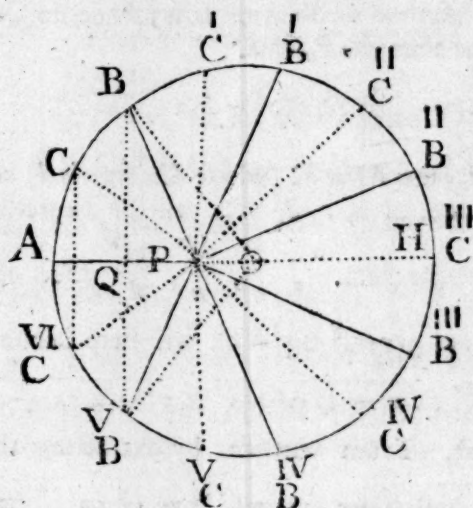
COROLLARY II.

316. But, if $k = -1$ (or the middle Term be $+ 2r^n x^n$) then the Arch R being $= \frac{P}{2}$, the Values of b, c, d , ∴c. will, here, be defined by the Co-fines of the

the Arcs $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$, &c. and our Equation, by taking the Root, as above, will become $r^n + x^n = \sqrt[n]{r^2 - 2brx + x^2}^{\frac{1}{2}} \times \sqrt[n]{r^2 - 2crx + x^2}^{\frac{1}{2}} (n)$.

SCHOLIUM.

317. From the two preceding Corollaries, the Demonstration of that remarkable Property of the Circle given, and apply'd to finding a vast Number of Fluents, in *Cotes's Harmonia Mensurarum*, is very easily, and naturally, deduced.



For, let the Periphery of the

Circle $ABB\&c.$ whose Radius is expressed by r , be divided into as many equal

Parts $AB, B\overset{I}{B}, B\overset{II}{B}, \&c.$ as there are Units in the given Integer n ; so that

$AB, A\overset{I}{B}, A\overset{II}{B},$

$\&c.$ may respectively exhibit the Values of the foresaid

Arcs $\frac{P}{n}, \frac{2P}{n}, \frac{3P}{n} \&c.$ (*vid. Corol. 1.*) Moreover, let

OQ be the Co-sine of the first of them; and, in the Radius OA (produced if necessary) let there be taken $OP=x$; and let $OB, QB, PB, \&c. \&c.$ be drawn:

Then, the Co-sine of the Angle $AOB (= \frac{P}{n})$ to the

by resolving them into more simple ones.

353

the Radius r , being expressed by c (*vid. Corol. 1.*) it will be $1 : c :: r (OB) : OQ = cr$: Whence $PB^2 (= OB^2 + OP^2 - 2OQ \times OP) = r^2 + x^2 - 2crx = r^2 - 2crx + x^2$.

By the very same Argument PB'^2 is $= r^2 - 2drx + x^2$, &c. &c. Therefore, because $r^n \propto x^n = \overline{r^2 - 2brx + x^2}^{\frac{1}{2}} \times \overline{r^2 - 2crx + x^2}^{\frac{1}{2}} \times \overline{r^2 - 2drx + x^2}^{\frac{1}{2}} (n)$, by *Corol. 1.* it follows that their Equals, $AO^n \propto OP^n$ and $PA \times PB \times PB' \times PB'' \times \dots$ must be equal likewise: Which is the first Part of the Theorem above hinted at.

After the same Manner, if the Arcs AC , AC' , AC'' , AC''' be taken respectively equal to $\frac{P}{2n}$, $\frac{3P}{2n}$, $\frac{5P}{2n}$ &c. it will appear (*from Corol. 2.*) that $AO^n + PO^n$ is $= PC \times PC' \times PC'' (n)$ Which is the latter Part of the same Theorem.

Hence (by the Bye) all the Roots of the Equation $x^n = r^n$ are very readily found: For, since $AO^n \propto PO^n = PA \times PB \times PB' \times PB'' \times \dots$ where the second Factor and the last, the third and the last but one, &c. are respectively equal to each other, it is evident that $AO^n \propto PO^n (r^n \propto x^n)$ is also $= PA \times PB^2 \times PB'^2 \times PB''^2 \times \dots = \overline{r \propto x} \times \overline{r^2 - 2crx + x^2} \times \overline{r^2 - 2drx + x^2} \times \dots$ &c.

Whence, $x^n \propto r^n$ being $= 0$, it follows that $r \propto x \times \overline{r^2 - 2crx + x^2} \times \dots$ is $= 0$: From which, by extracting the Roots out of the Equations $r \propto x = 0$, $r^2 - 2crx + x^2 = 0$, &c. we get r , $r \times c + \sqrt{c^2 - 1}$, $r \times c - \sqrt{c^2 - 1}$, $r \times d + \sqrt{d^2 - 1}$, &c. A a

ℳc. for the several Roots of the Equation $x^n = r^n$; whereof the first, only, is possible.

By the same Way of proceeding all the Roots of the Equation, $x^n + r^n = 0$, will also be found: For, seeing $x^n + r^n$ is $= \sqrt[n]{r^2 - 2brx + x^2}^{\frac{1}{2}} \times \sqrt[n]{r^2 - 2crx + x^2}^{\frac{1}{2}} \mathcal{E}c.$

($= PC \times PC' \times PC'' \mathcal{E}c.$) where the first Factor and the last, the second and the last but one, ℳc. are respectively equal to each other, it is plain that $x^n + r^n$ is likewise $= \sqrt[n]{r^2 - 2brx + x^2} \times \sqrt[n]{r^2 - 2crx + x^2} \mathcal{E}c.$ and consequently $x = r \times b \pm \sqrt[n]{b^2 - 1} \mathcal{E}c. \mathcal{E}c.$ Where the Roots are all impossible; except the last, when their Number (n) is odd.

LEMMA IV.

318. Supposing every thing to remain as in the preceding Lemma, and that k, b, c, d &c. denote the Sines of the Arcs $R, \frac{R}{n}, \frac{P+R}{n}, \frac{2P+R}{n} \mathcal{E}c.$ (whose Co-sines are $k, b, c, d,$ &c.) then, I say, the Fraction

$$\frac{\overset{\cdot}{n} k r^n x^n}{r^{2n} - 2k r^n x^n + x^{2n}} \text{ is equal to } \frac{\overset{\cdot}{b} r x}{r^2 - 2b r x + x^2} + \frac{\overset{\cdot}{c} r x}{r^2 - 2c r x + x^2} + \frac{\overset{\cdot}{d} r x}{r^2 - 2d r x + x^2} \mathcal{E}c.$$

For, since $r^{10} - 2kr^5x^5 + x^{10} (r^{2n} - 2kr^n x^n + x^{2n})$ is $= \sqrt[n]{r^2 - 2arx + x^2} \times \sqrt[n]{r^8 + Ar^7x + Br^6x^2 + Cr^5x^3 + Dr^4x^4 + Cr^3x^5 + Br^2x^6 + Arx^7 + x^8}$ (by the foresaid Lemma) and it is also proved that $A = \frac{\text{Sin. } 2\mathcal{Q}}{s}, B = \frac{\text{Sin. } 3\mathcal{Q}}{s},$

I

C =

by resolving them into more simple ones. 355

$C = \frac{\text{Sin. } 4Q}{s}$ &c. it is evident, therefore, that

$$\frac{r^{10} - 2kr^5x^5 + x^{10}}{r^2 - 2arx + x^2} (=r^8 + Ar^7x + Br^6x^2 \text{ \&c.}) \text{ is } =r^8 +$$

$$\frac{\text{Sin. } 2Q}{s} \times r^7x \text{ \&c. and consequently } \frac{s \times r^{10} - 2kr^5x^5 + x^{10}}{r^2 - 2arx + x^2}$$

$$= \text{Sin. } Q \times r^8 + \text{Sin. } 2Q \times r^7x + \text{Sin. } 3Q \times r^6x^2 + \text{Sin. } 4Q \times r^5x^3 + \text{Sin. } 5Q \times r^4x^4 + \text{Sin. } 4Q \times r^3x^5 \text{ \&c. In}$$

which Equation, for a and s , let their several respec-

tive Values $b, c, d, \text{ \&c.}$ and $b', c', d', \text{ \&c.}$ be, succe-

sively, substituted; and let the corresponding Arcs $\frac{R}{n},$

$$\frac{P+R}{n}, \frac{2P+R}{n} \text{ \&c. be represented by } Q, Q', Q'', \text{ \&c.}$$

then we shall have

$$\frac{b \times r^{10} - 2kr^5x^5 + x^{10}}{r^2 - 2brx + x^2} = \text{Sin. } Q \times r^8 + \text{Sin. } 2Q \times r^7x \text{ \&c.}$$

$$\frac{c \times r^{10} - 2kr^5x^5 + x^{10}}{r^2 - 2crx + x^2} = \text{Sin. } Q' \times r^8 + \text{Sin. } 2Q' \times r^7x \text{ \&c.}$$

&c.

&c.

Which Equations, added all together, give

$$\frac{r^{10} - 2kr^5x^5 + x^{10}}{r^2 - 2krx + x^2} \times \frac{b}{r^2 - 2brx + x^2} + \frac{c}{r^2 - 2crx + x^2} +$$

$$\frac{d}{r^2 - 2drx + x^2} \text{ \&c.}$$

$$\begin{aligned}
 &= \left\{ \begin{array}{l} \text{Sin. } \frac{p}{n} \\ \text{Sin. } \frac{2p}{n} \\ \text{Sin. } \frac{3p}{n} \end{array} \right\} \times r^3 + \left\{ \begin{array}{l} \text{Sin. } \frac{2p}{n} \\ \text{Sin. } \frac{2p}{n} \\ \text{Sin. } \frac{2p}{n} \end{array} \right\} \times r^2 + \dots + \left\{ \begin{array}{l} \text{Sin. } \frac{n}{n} \\ \text{Sin. } \frac{n}{n} \\ \text{Sin. } \frac{n}{n} \end{array} \right\} \times r^4 + \text{Ec.}
 \end{aligned}$$

But the Sines of the first Column, being those of an arithmetical Progression (whose common Difference is $\frac{p}{n}$) by which the whole Periphery is divided into n (5) equal Parts, their Sum will therefore, it is well known, be equal to Nothing; or all the negative ones equal to all the positive ones.

The

by resolving them into more simple ones. 357

The same is also true with regard to the Sines of the second Column; whose Arcs $\frac{2R}{n}$, $\frac{2P+2R}{n}$, $\frac{4P+2R}{n}$

&c. (having $\frac{2P}{n}$ for their common Difference) divide

the Periphery (twice taken) into the same Number (n) of equal Parts. But the Sines of the middle Column (which is the last above exhibited) will not vanish, as

all the rest do: For, nQ being $=R$, $nQ=P+R$, $nQ=2P+R$, &c. the common Difference will here be equal to (P) the whole Periphery; and therefore, every Arch terminating in the same Point with the first, the Circle will, in this Case, remain undivided, and the

Sine of each be equal to (k) the Sine of the first.

Hence, our Equation is reduced to $r^{10}-2kr^5x^5+x^{10} \times$

$$\frac{b}{r^2-2brx+x^2} + \frac{c}{r^2-2crx+x^2} \&c. = 5kr^4x^4; \text{ which}$$

divided by $r^{10}-2kr^5x^5+x^{10}$, and multiply'd by rx , gives

$$\frac{brx}{r^2-2brx+x^2} + \frac{crx}{r^2-2crx+x^2} + \frac{drx}{r^2-2drx+x^2} \&c. =$$

$$\frac{5kr^5x^5}{r^{10}-2kr^5x^5+x^{10}} = \frac{nkr^n x^n}{r^{2n}-2kr^n x^n+x^{2n}}. \quad Q. E. D.$$

The same otherwise.

319. Since $r^{2n}-2kr^n x^n+x^{2n}$ is $= \frac{r^2-2brx+x^2}{r^2-2crx+x^2} \times \frac{r^2-2drx+x^2}{r^2-2drx+x^2} (n)$ by Lemma 3. it is

evident that, $\text{Log. } r^{2n}-2kr^n x^n+x^{2n} = \text{Log.}$

$$\frac{r^2-2brx+x^2}{r^2-2crx+x^2} + \text{Log. } \frac{r^2-2drx+x^2}{r^2-2drx+x^2}$$

(n). And, as this Equation holds universally, let k and x be what they will (which two Quantities may be supposed to flow independently of each other) let the

A a 3

Fluxion

Fluxion of the *whole* Equation be taken, making k va-

riable (and x constant); which gives $\frac{-2\dot{k}r^n x^n}{r^{2n} - 2\dot{k}r^n x^n + x^{2n}}$

$$= \frac{-2\dot{b}rx}{r^2 - 2\dot{b}rx + x^2} - \frac{2\dot{c}rx}{r^2 - 2\dot{c}rx + x^2} - \frac{2\dot{d}rx}{r^2 - 2\dot{d}rx + x^2}$$

* Art. 126. (n)*. But, $k, b, c, d, \&c.$ are the Cosines of the Arcs $R,$

$\frac{R}{n}, \frac{R+P}{n}, \frac{R+2P}{n}, \&c.$ (whereof the corresponding

Sines are $\dot{k}, \dot{b}, \dot{c}, \&c.$) therefore, the Fluxion of the first of these Arcs being denoted by $\dot{R}$, the Fluxion of each of the rest will be expressed by $\frac{\dot{R}}{n}$; And so (the

Fluxion of the Co-sine of an Arch being equal to the Fluxion of the Arch itself drawn into its Sine, apply'd to

* Art. 142. Radius *) it follows that $\dot{k} = \dot{R}\dot{k}, \dot{b} = \frac{\dot{R}}{n} \times \dot{b}, \dot{c} =$

$\frac{\dot{R}}{n} \times \dot{c}, \&c.$ Which Values being substituted in the

foregoing Equation, and the whole divided by $\frac{-2\dot{R}}{n},$

$$\text{we have } \frac{n\dot{k}r^n x^n}{r^{2n} - 2\dot{k}r^n x^n + x^{2n}} = \frac{\dot{b}rx}{r^2 - 2\dot{b}rx + x^2} +$$

$$\frac{\dot{c}rx}{r^2 - 2\dot{c}rx + x^2} + \frac{\dot{d}rx}{r^2 - 2\dot{d}rx + x^2} \quad (n).$$

LEMMA V.

320. To determine the Series, arising from the Division of Unity by a Trinomial, $x^2 - 2arx + r^2$; and to exhibit the Remainder after any given Number (v) of Terms in the Quotient.

Let $x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + Dr^4x^{-6}$ represent the required Quotient continued to 5 Terms ($v,$

by resolving them into more simple ones.

359

(*v*, to render the Process the more obvious, being first expounded by that Number) and let $Er^5x^{-5} + Fr^6x^{-6}$

be the Remainder. Then, because $\frac{1}{x^2 - 2arx + r^2}$ is $=$
 $\frac{x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + Dr^4x^{-6} + Er^5x^{-5} + Fr^6x^{-6}}{x^2 - 2arx + r^2}$, we shall, by reducing the whole Equation to one Denomination, have

$$\begin{array}{c} 1 + Arx^{-1} + Br^2x^{-2} + Cr^3x^{-3} + Dr^4x^{-4} + Er^5x^{-5} + Fr^6x^{-6} \\ - 2arx^{-1} - 2aAr^2x^{-2} - 2aBr^3x^{-3} - 2aCr^4x^{-4} - 2aDr^5x^{-5} \\ + r^2x^{-2} + Ar^3x^{-3} + Br^4x^{-4} + Cr^5x^{-5} + Dr^6x^{-6} \end{array} = 0$$

Whence $A=2a$, $B=2aA-1$, $C=2aB-A$, $D=2aC-B$, $E=2aD-C$, and $F=-D$.

A a 4

There-

Therefore, if $\mathcal{Q}$ be now put for the Arch whose Radius is 1 and Co-sine a , and there be taken $S = \text{Sin.}$

$\mathcal{Q}$, $\dot{S} = \text{Sin. } 2\mathcal{Q}$, $\ddot{S} = \text{Sin. } 3\mathcal{Q}$, &c. we shall, also, have

$$A(2a) = \frac{\dot{S}}{S}, B = \frac{\ddot{S}}{S}, C = \frac{\dddot{S}}{S}, D = \frac{\text{Sin. } 5\mathcal{Q}}{S}, E = \frac{\text{Sin. } 6\mathcal{Q}}{S}$$

$$= \frac{\text{Sin. } 6\mathcal{Q}}{S}, F(-D) = -\frac{\text{Sin. } 5\mathcal{Q}}{S} \text{ (by Corol. to}$$

Lem. 1.) And consequently $\frac{1}{x^2 - 2arx + r^2} =$

$$\frac{Sx^{-2} + \dot{S}rx^{-3} + \ddot{S}r^2x^{-4} + \dddot{S}r^3x^{-5} + \text{Sin. } 5\mathcal{Q}r^4x^{-6}}{S} +$$

$$\frac{\text{Sin. } 6\mathcal{Q}r^5x^{-7} - \text{Sin. } 5\mathcal{Q}r^6x^{-8}}{Sx^2 - 2arx + r^2}, \text{ Whence, univer-}$$

$$\text{sally, } \frac{1}{x^2 - 2arx + r^2} =$$

$$\frac{Sx^{-2} + \dot{S}rx^{-3} + \ddot{S}r^2x^{-4} + \dddot{S}r^3x^{-5} \text{ \&c. (to } v \text{ Terms)}}{S} +$$

$$\frac{\text{Sin. } v+1\mathcal{Q} \times r^v x^{-v-1} - \text{Sin. } v\mathcal{Q} \times r^{v+1} x^{-v-2}}{Sx^2 - 2arx + r^2}. \text{ Which}$$

last Equation (tho' obvious enough from the preceding one) may be investigated in a general Manner (if re-

quired) by assuming $x^{-2} + Arx^{-3} + Br^2x^{-4} + Cr^3x^{-5} + \dots + dr^{v-2}x^{-v} + er^{v-1}x^{-v-1} +$

$$\frac{fr^v x^{-v} + gr^{v+1} x^{-v-1}}{x^2 - 2arx + r^2} = \frac{1}{x^2 - 2arx + r^2}, \text{ and proceed-}$$

ing as above: By which Means you will find $A=2a$,

$$B=2aA-1, \text{ \&c. } f=2ae-d = \frac{\text{Sin. } v+1\mathcal{Q}}{S}, \text{ and } g$$

$$(-e) = -\frac{\text{Sin. } v\mathcal{Q}}{S}. \text{ And thus may the third}$$

Lemma

by resolving them into more simple ones. 361

Lemma be made out, if any Objection, or Difficulty, should arise about its being general.

COROLLARY.

321. If, in the given Trinomial $x^2 - 2arx + r^2$, we suppose r^2 , instead of x^2 , to be the leading Term whereby the Quotient is produced; then, since r and x are affected exactly alike; we shall, by writing r for

x , and x for r , have $\frac{1}{r^2 - 2arx + x^2} =$

$$\frac{Sr^{-2} + \dot{S}xr^{-3} + \ddot{S}x^2r^{-4} (v)}{S} +$$

$$\frac{\text{Sin. } v+1 \times 2 \times x^v r^{-v} - \text{Sin. } v \times x^{v+1} \times r^{-v-1}}{S \times r^2 - 2arx + x^2}$$

PROB. I.

322. To find the Fluents of $\frac{x}{rr - 2arx + xx}$, together with

that of $\frac{xx}{rr - 2arx + xx}$.

Let ABM &c. be a Circle whose Radius OA (or OM) is r , and let the Angle AOB be such, that its Co-sine, to the Radius 1, may be equal to a ; or so, that OQ (supposing BQ perpendicular to OA) may be $= ar$: Moreover let s denote the Sine of the said Angle AOB, cor-



responding to the Co-sine a , and let OP (considered as variable by the Motion of P along OA) express the Value of x : Then, PB^2 ($OB^2 + OP^2 - 2OQ \times OP$) $= rr - 2arx + xx$: And the Fluxion of the Measure of the Angle QBP (Radius being Unity) will be represented

sented by $\frac{BQ \times \text{Flux. } QP}{BP^2}$ (vid. Art. 142.) or by

$\frac{rs \times -\dot{x}}{rr - 2arx + xx}$; and consequently that of OBP , by

$\frac{rs \dot{x}}{rr - 2arx + xx}$: Whence it is evident that the Fluent of

$\frac{\dot{x}}{rr - 2arx + xx}$ (contemporaneous with x) is truly ex-

pressed by $\frac{r}{rs} \times OBP$.

Again, since $\frac{x\dot{x}}{rr - 2arx + xx}$ may be transformed to

$\frac{-ar\dot{x} + x\dot{x}}{rr - 2arx + xx} + \frac{ar\dot{x}}{rr - 2arx + xx}$; where the Fluent of

• Art. 126. the former Part is $= \frac{1}{2}$ hyp. Log. $\frac{rr - 2arx + xx}{rr} =$

$\frac{1}{2}$ hyp. Log. $\frac{PB^2}{OB^2} = \text{hyp. Log. } \frac{PB}{OB}$; and that of the latter

Part $= \frac{a}{s} \times OBP$; it appears that the Fluent of

$\frac{x\dot{x}}{rr - 2arx + xx}$ is truly expounded by $\text{hyp. Log. } \frac{PB}{OB} + \frac{a}{s} \times OBP$. Q. E. I.

COROLLARY

323. Since, $PB : PO :: \text{Sin. } BOP (s) : \text{Sin. } OBP =$

$\frac{sx}{\sqrt{rr - 2arx + xx}}$; it follows, if the hyperbolical Lo-

garithm of $\frac{\sqrt{r^2 - 2arx + xx}}{r}$, be represented by M , and

the Arch, whose Sine is $\frac{sx}{\sqrt{rr - 2arx + xx}}$ and Radius

Unity,

by resolving them into more simple ones.

363

Unity by N , that the Fluents of $\frac{x}{rr-2arx+xx}$ and $\frac{xx}{rr-2arx+xx}$ will be expressed by $\frac{N}{rs}$ and $M + \frac{aN}{s}$ respectively,

PROB. II.

324. To determine the Fluent of $\frac{x^m \dot{x}}{x^2-2arx+r^2}$; supposing m any whole positive Number, and a less than Unity.

Let every thing remain as in Lemma 5. and then, if the Equation there brought out be multiply'd by $x^m \dot{x}$, and v at the same time be expounded by $m-1$, we shall

$$\text{get } \frac{x^m \dot{x}}{x^2-2arx+r^2} = \frac{Sx^{m-2}\dot{x} + \dot{S}rx^{m-3}\dot{x} + \dot{S}r^2x^{m-4}\dot{x}}{S}$$

$$(m-1) + \frac{\overline{\text{Sin. } m\mathcal{Q} \times r^{m-1}xx} - \overline{\text{Sin. } m-1 \times \mathcal{Q} \times r^m \dot{x}}}{S \times xx - 2arx + rr}$$

Whose Fluent will therefore be given by the preceding Proposition: For, supposing the Values of M and N to be as there specified, the Fluent of the last Term

$$\left(\frac{\overline{\text{Sin. } m\mathcal{Q} \times r^{m-1}xx} - \overline{\text{Sin. } m-1 \times \mathcal{Q} \times r^m \dot{x}}}{S \times xx - 2arx + rr} \right) \text{ will, it}$$

is manifest*, be expressed by $\frac{1}{S}$ into $\overline{\text{Sin. } m\mathcal{Q} \times r^{m-1} \times}$ * Art. 323.

$$M + \frac{aN}{S} - \overline{\text{Sin. } m-1 \times \mathcal{Q} \times r^m \times} \times \frac{N}{rS} = \frac{r^{m-1}}{S} \text{ into}$$

$$\overline{\text{Sin. } m\mathcal{Q} \times M} + \frac{\overline{\text{Sin. } m\mathcal{Q} \times a}}{S} - \frac{\overline{\text{Sin. } m-1 \times \mathcal{Q}}}{S} \times N$$

$$= \frac{r^{m-1}}{S} \text{ into } \overline{\text{Sin. } m\mathcal{Q} \times M} + \overline{\text{Co-f. } m\mathcal{Q} \times N} \text{ (by Lem. 2.)}$$

Case

Case 1.) To which adding the Fluent of the preceding Series,

$$\text{there results } \frac{1}{S} \times \frac{Sx^{m-1}}{m-1} + \frac{\dot{S}rx^{m-2}}{m-2} + \frac{\ddot{S}r^2x^{m-3}}{m-3} (m-1) \\ + \frac{r^{m-1}}{S} \times \frac{\text{Sin. } m\mathcal{Q} \times M + \text{Co-f. } m\mathcal{Q} \times N}{\mathcal{Q}. E. I.}$$

COROLLARY.

325. Hence, the Fluent of $\frac{-ax^m \dot{x} + rx^{m-1} \dot{x}}{xx - 2arx + rr}$ may be deduced: For, by writing $m-1$, instead of m , the Fluent of $\frac{x^{m-1} \dot{x}}{xx - 2arx + rr}$ will be found $= \frac{1}{S} \times$

$$\frac{Sx^{m-2}}{m-2} + \frac{\dot{S}rx^{m-3}}{m-3} + \frac{\ddot{S}r^2x^{m-4}}{m-4} (m-2) + \frac{r^{m-2}}{S} \times \\ \text{Sin. } m-1 \times \mathcal{Q} \times M + \text{Co-f. } m-1 \times \mathcal{Q} \times N: \text{ Which Flu-} \\ \text{ent being multiply'd by } r, \text{ and that of } \frac{x^m \dot{x}}{xx - 2arx + rr} \\ \text{(given above) by } -a, \text{ we shall, when the homologous}$$

Terms are united, have $\frac{1}{S} \times -aS \times \frac{x^{m-1}}{m-1} - a\dot{S} - S \times$

$$\frac{rx^{m-2}}{m-2} - a\ddot{S} - \dot{S} \times \frac{r^2x^{m-3}}{m-3} (m-1) + \frac{r^{m-1}}{S} \text{ into } - \\ \text{Sin. } m\mathcal{Q} \times a - \text{Sin. } m-1 \times \mathcal{Q} \times M - \text{Co-f. } m\mathcal{Q} \times a - \\ \text{Co-f. } m-1 \times \mathcal{Q} \times N, \text{ for the true Fluent of the Quan-} \\ \text{tity propounded.}$$

$$\text{But (by Case 1. Lem. 2.) } \frac{a\dot{S} - S}{S} (= \\ \frac{\text{Sin. } 2\mathcal{Q} \times a - \text{Sin } \mathcal{Q} \times \text{Rad.}}{\text{Sin. } \mathcal{Q}}) = \text{Co-f. } 2\mathcal{Q}: \text{ Also}$$

by resolving them into more simple ones. 365

$$\frac{a\dot{S}-\dot{S}}{S} \left(\frac{\text{Sin. } 3\dot{Q} \times a - \text{Sin. } 2\dot{Q} \times \text{Rad.}}{S} \right) = \text{Co-f. } 3\dot{Q}$$

Ec. And, by Case 2. of the same Lemma,

$$\frac{\text{Co-f. } m-1 \times \dot{Q} - \text{Co-f. } m\dot{Q} \times a}{S} = \text{Sin. } m\dot{Q}: \text{ Whence,}$$

by substituting these Values, our Fluent is reduced to

$$\begin{aligned} & -\text{Co-f. } \dot{Q} \times \frac{x^{m-1}}{m-1} - \text{Co-f. } 2\dot{Q} \times \frac{r x^{m-2}}{m-2} - \text{Co-f. } 3\dot{Q} \times \\ & \frac{r^2 x^{m-3}}{m-3} - \text{Co-f. } 4\dot{Q} \times \frac{r^3 x^{m-4}}{m-4} (m-1) - r^{m-1} \times \\ & \frac{\text{Co-f. } m\dot{Q} \times M - \text{Sin. } M\dot{Q} \times N}{S} \end{aligned}$$

PROB. III.

326. To determine the Fluent of $\frac{x^{-m}\dot{x}}{r^2-2arx+x^2}$; under the Restrictions specified in the preceding Problem.

If the Equation in Art. 321. be multiply'd by $x^{-m}\dot{x}$, and v at the same time be expounded by m , we

$$\text{shall have } \frac{x^{-m}\dot{x}}{r^2-2arx+x^2} =$$

$$\frac{Sr^{-2}x^{-m}\dot{x} + Sr^{-3}x^{1-m}\dot{x} + Sr^{-4}x^{2-m}\dot{x}}{S} (m) +$$

$$\frac{r^{-m-1}}{S} \times \frac{\text{Sin. } m+1 \times \dot{Q} \times r\dot{x} - \text{Sin. } m\dot{Q} \times x\dot{x}}{r^2-2arx+x^2} :$$

Where, the Fluent of the last Term being $\frac{r^{-m-1}}{S} \times$

$$\text{Sin. } m+1 \times \dot{Q} \times \frac{N}{S} - \text{Sin. } m\dot{Q} \times M + \frac{aN}{S} * = * \text{ Art. 323}$$

$$\frac{r^{-m-1}}{S} \text{ into } -\text{Sin. } m\dot{Q} \times M +$$

366 *Of the Fluents of Rational Fractions,*

$$\frac{\text{Sin. } m+1 \times \overline{Q} - \text{Sin. } m \overline{Q} \times a}{S} \times N = \frac{r^{m-1}}{S} \times$$

$\frac{-\text{Sin. } m \overline{Q} \times M + \text{Co-f. } m \overline{Q} \times N}{S}$ (by Case 3. Lem. 2.)
it follows that the Fluent of the whole Expression, or
the Quantity sought, will be truly expressed by

$$\frac{1}{S} \times \frac{S r^{-2} x^{1-m}}{1-m} + \frac{S r^{-3} x^{2-m}}{2-m} + \text{Ec. or its Equal}$$

$$\frac{-1}{S} \times \frac{S x^{1-m}}{m-1.r^2} + \frac{S x^{2-m}}{m-2.r^3} + \frac{S x^{3-m}}{m-3.r^4} (m) +$$

$$\frac{1}{S r^{m+1}} \times \overline{\text{Co-sin. } m \overline{Q} \times N - \text{Sin. } m \overline{Q} \times M.}$$

PROB. IV.

327. To find the Fluent of $\frac{x^{m-1} \dot{x}}{r^n + x^n}$; m and n being

any whole positive Numbers, whereof the former does
not exceed the latter.

Let $b, c, d, \text{Ec.}$ denote the Co-sines of the Arcs
 $\frac{360^\circ}{2n}, \frac{3 \times 360^\circ}{2n}, \frac{5 \times 360^\circ}{2n}, \text{Ec.}$ (Radius being Unity)

Then (by Corol. 2. Lem. 3.) we shall have $r^n + x^n =$
 $\sqrt{rr-2brx+xx}^{\frac{1}{2}} \times \sqrt{rr-2crx+xx}^{\frac{1}{2}} \times \sqrt{rr-2drx+xx}^{\frac{1}{2}}$

(n). Whence $\text{Log. } r^n + x^n = \frac{1}{2} \text{Log. } rr-2brx+xx +$
 $\frac{1}{2} \text{Log. } rr-2crx+xx + \frac{1}{2} \text{Log. } rr-2drx+xx$ (n)
and, consequently, by taking the Fluxion, on both

Sides, $\frac{n x^{n-1} \dot{x}}{r^n + x^n} = \frac{xx-br\dot{x}}{xx-2brx+rr} + \frac{xx-cr\dot{x}}{xx-2crx+rr} +$

• Art. 126. $\frac{xx-dr\dot{x}}{xx-2drx+rr} * (n)$; which last Equation, multiply'd by

$$\frac{x}{\dot{x}}, \text{ gives } \frac{n x^n}{r^n + x^n} = \frac{xx-brx}{xx-2brx+rr} + \frac{xx-crx}{xx-2crx+rr}$$

by resolving them into more simple ones. 367

+ $\frac{xx-drx}{xx-2drx+rr}$ (n). Let each Side hereof be now subtracted from n (or, which comes to the same thing, let $\frac{nx^n}{r^n+x^n}$ be taken from n , and each of the (n)

Terms on the other Side, from Unity) then we shall have $\frac{nr^n}{r^n+x^n} = \frac{-brx+rr}{xx-2brx+rr} + \frac{-crx+rr}{xx-2crx+rr}$

+ $\frac{-drx+rr}{xx-2drx+rr}$ (n): Which multiply'd by $\frac{x^{m-1}\dot{x}}{r}$,

gives $\frac{nr^{n-1} \times x^{m-1}\dot{x}}{r^n+x^n} = \frac{-bx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2brx+rr} +$

$\frac{-cx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2crx+rr}$ &c.

But now, to determine the Fluent hereof, let the several Arcs $\left(\frac{180^\circ}{n}, \frac{3 \times 180^\circ}{n}, \frac{5 \times 180^\circ}{n}\right.$ &c.) above

specified, be denoted by $\mathcal{Q}, \mathcal{Q}', \mathcal{Q}'', \mathcal{Q}''',$ &c. respectively; also let $N, N', N'',$ &c. express the Measures of the Angles whose Sines are $\frac{x \times \text{Sin. } \mathcal{Q}}{\sqrt{rr-2brx+xx}},$

$\frac{x \times \text{Sin. } \mathcal{Q}'}{\sqrt{xx-2crx+rr}}, \frac{x \times \text{Sin. } \mathcal{Q}''}{\sqrt{xx-2drx+rr}}$ &c. and $M, M',$

$M'',$ &c. the hyperbolic Logarithms of $\frac{\sqrt{xx-2brx+rr}}{r},$

$\frac{\sqrt{xx-2crx+rr}}{r}, \frac{\sqrt{xx-2drx+rr}}{r}$ &c. Then (by

Corol. to Prob. 2.) the Fluent of the first Term,

$\frac{-bx^m\dot{x}+rx^{m-1}\dot{x}}{xx-2brx+rr}$ (expounding a by b) comes out

$$- \text{Co-f. } 2 \times \frac{x^{m-1}}{m-1} - \text{Co-f. } 2 \times \frac{rx^{m-2}}{m-2} - \text{Co-f. } 3 \times \frac{r^2 x^{m-3}}{m-3} (m-1) + r^{m-1} \text{ into } \frac{\text{Sin. } m \times N - \text{Co-f. } m \times M}{M}$$

In the same Manner, by writing c for a , $\dot{Q}$ for Q , $\dot{M}$ for M , and $\dot{N}$ for N) the Fluent of the second Term, $\frac{-cx^m \dot{x} + rx^{m-1} \dot{x}}{xx - 2crx + rr}$, is found $= - \text{Co-f. } \dot{Q} \times \frac{x^{m-1}}{m-1} - \text{Co-f. } 2 \dot{Q} \times \frac{rx^{m-2}}{m-2} \text{ \&c. \&c.}$

Therefore the Fluent of the whole Expression, by collecting the homologous Terms, appears to be

$$\left\{ \begin{array}{l} \text{Co-f. } 2 \dot{Q} \\ \text{Co-f. } 2 \dot{Q} \\ \text{Co-f. } 2 \dot{Q} \\ \text{Co-f. } 2 \dot{Q} \\ \text{\&c.} \end{array} \right\} \times \frac{x^{m-1}}{m-1} - \left\{ \begin{array}{l} \text{Co-f. } 2.2 \dot{Q} \\ \text{Co-f. } 2.2 \dot{Q} \\ \text{Co-f. } 2.2 \dot{Q} \\ \text{Co-f. } 2.2 \dot{Q} \\ \text{\&c.} \end{array} \right\} \times \frac{rx^{m-2}}{m-2} - \left\{ \begin{array}{l} \text{Co-f. } 3.3 \dot{Q} \\ \text{Co-f. } 3.3 \dot{Q} \\ \text{Co-f. } 3.3 \dot{Q} \\ \text{Co-f. } 3.3 \dot{Q} \\ \text{\&c.} \end{array} \right\} \times \frac{r^2 x^{m-3}}{m-3} \text{\&c.}$$

by resolving them into more simple ones.

369

$$+ r^{m-1} \times \left\{ \begin{array}{l} \overline{\text{Sin. } m\mathcal{Q} \times N} - \overline{\text{Co-f. } m\mathcal{Q} \times M} \\ \overline{\text{Sin. } m'\mathcal{Q} \times N} - \overline{\text{Co-f. } m'\mathcal{Q} \times M} \\ \overline{\text{Sin. } m''\mathcal{Q} \times N} - \overline{\text{Co-f. } m''\mathcal{Q} \times M} \\ \overline{\text{Sin. } m'''\mathcal{Q} \times N} - \overline{\text{Co-f. } m'''\mathcal{Q} \times M} \\ \text{\textit{\textcircled{C}}c.} \quad \text{\textit{\textcircled{C}}c.} \end{array} \right.$$

But the Co-sines of the first Column being those of an arithmetical Progression $\left(\frac{180^\circ}{n}, \frac{3 \times 180^\circ}{n}, \frac{5 \times 180^\circ}{n} \right.$

$\text{\textit{\textcircled{C}}c.})$ whose common Difference is $\frac{360^\circ}{n}$, whereby the

whole Periphery is divided into n equal Parts (*vid. Art. 317.*) they will therefore destroy one another; since it is well known that, if the Periphery of any Circle be divided into any Number (n) of equal Parts, the negative Sines and Co-sines will be equal to the positive ones; which is self-evident when their Number is even.

Hence the Co-sines in the second and third Columns, $\text{\textit{\textcircled{C}}c.}$ will also destroy one another (*vid. Art. 318.*) But those of the last Column of all, as well as the Sines, having unequal Multipliers, must remain as above, and that Column, alone, (drawn into r^{m-1}) will be the true Fluent of $\frac{nr^{n-1} \times x^{m-1} \dot{x}}{r^n + x^n}$. Whence, putting $m\mathcal{Q}$

$(= m \times \frac{180^\circ}{n}) = R$, and dividing by nr^{n-1} , we

shall (because $\mathcal{Q} = 3\mathcal{Q}$, $\mathcal{Q}' = 5\mathcal{Q}$, $\mathcal{Q}'' = 7\mathcal{Q}$ $\text{\textit{\textcircled{C}}c.}$) have

$$\frac{x^{m-n}}{n} \times \left\{ \begin{array}{l} \overline{\text{Sin. } R \times N - \text{Co-f. } R \times M} \\ \overline{\text{Sin. } 3R \times N' - \text{Co-f. } 3R \times M'} \\ \overline{\text{Sin. } 5R \times N'' - \text{Co-f. } 5R \times M''} \\ \overline{\text{Sin. } 7R \times N''' - \text{Co-f. } 7R \times M'''} \\ \text{\textit{\text{Etc. (to } n \text{ Lines.)}} \end{array} \right\} = \text{Fluent of } \frac{x^{m-1} \dot{x}}{r^n + x^n}$$

Q. E. I.

COROLLARY.

328. Since the first and the last, the second and the last but one, &c. of the foregoing Quantities $x^2 - 2brx + rr$, $xx - 2crx + rr$, $xx - 2drx + rr$ &c. are respectively equal to each other (*vid. Art. 317.*) the corresponding Fluents, found above, will likewise be equal: And therefore the Fluent of

$\frac{x^{m-1} \dot{x}}{r^n + x^n}$ will, *also*, be expressed by

$$\frac{x^{m-n}}{n} \times \left\{ \begin{array}{l} \overline{\text{Sin. } R \times 2N - \text{Co-f. } R \times 2M} \\ \overline{\text{Sin. } 3R \times 2N' - \text{Co-f. } 3R \times 2M'} \\ \overline{\text{Sin. } 5R \times 2N'' - \text{Co-f. } 5R \times 2M''} \\ \text{\textit{\text{Etc.}} \qquad \qquad \qquad \text{\textit{\text{Etc.}}} \end{array} \right.$$

The Number of Lines to be thus taken being

$$= \frac{1}{2}n, \text{ when } n \text{ is even; but, otherwise, } = \frac{n+1}{2}; \text{ in}$$

which last Case, the Logarithm, &c. in the last Line, must be taken only once, instead of twice; being that

of $\frac{r+x}{r}$ (*vid. Art. 317.*)

PROB.

by resolving them into more simple ones. 371

PROB. V.

329. To find the Fluent of $\frac{x^{m-1} \dot{x}}{r^n - x^n}$; m and n being as in the preceding Problem.

If $b, c, d, \&c.$ be taken to denote the Co-sines of the Arcs $\frac{0}{n}, \frac{360^\circ}{n}, \frac{2 \times 360^\circ}{n} \&c.$ to n Terms, it will appear (from Corol. 1. to Lem. 3.) that $r^n - x^n$ is =

$\sqrt[n]{rr-2brx+xx}^{\frac{1}{2}} \times \sqrt[n]{rr-2crx+xx}^{\frac{1}{2}} \times \sqrt[n]{rr-2drx+xx}^{\frac{1}{2}} \&c.$ (n). From whence, by following the Method of the

last Problem, we also have $\frac{nr^{n-1} \times x^{m-1} \dot{x}}{r^n - x^n} =$

$$\frac{-bx^m \dot{x} + rx^{m-1} \dot{x}}{xx-2brx+rr} + \frac{-cx^m \dot{x} + rx^{m-1} \dot{x}}{xx-2crx+rr} \&c.$$

Which Fluxion having exactly the same Form with that in the preceding Problem, its Fluent will also be expressed in the very same Manner; that is, by

$$r^{m-1} \times \begin{cases} \text{Sin. } m\mathcal{Q} \times N - \text{Co-f. } m\mathcal{Q} \times M \\ \text{Sin. } m'\mathcal{Q} \times N' - \text{Co-f. } m'\mathcal{Q} \times M' \\ \text{Sin. } m''\mathcal{Q} \times N'' - \text{Co-f. } m''\mathcal{Q} \times M'' \end{cases}$$

(&c. to n Lines.)

Only $\mathcal{Q}, \mathcal{Q}', \mathcal{Q}'' \&c.$ must here stand for $\frac{0}{n}, \frac{360^\circ}{n}, \frac{2 \times 360^\circ}{n}, \frac{3 \times 360^\circ}{n} \&c.$ (instead of $\frac{180^\circ}{n}, \frac{3 \times 180^\circ}{n}, \frac{5 \times 180^\circ}{n} \&c.$)

372 *Of the Fluents of Rational Fractions,*

Therefore, since the multiple Arcs $m\mathcal{Q}$, $m\mathcal{Q}$, $m\mathcal{Q}$ &c. are, in this Case, equal to 0, $m \times \frac{360^\circ}{n}$, $2m \times \frac{360^\circ}{n}$, $3m \times \frac{360^\circ}{n}$ &c. (whereof the Sine of the first is = 0, and its Co-sine = Unity) we shall, by putting $R = m \times \frac{360^\circ}{n}$, and dividing the foresaid Fluent by nr^{n-1} , have

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} \text{.....} \text{.....} M \\ \text{Sin. } R \times \dot{N} - \text{Co-f. } R \times \dot{M} \\ \text{Sin. } 2R \times \ddot{N} - \text{Co-f. } 2R \times \ddot{M} \\ \text{Sin. } 3R \times \ddot{N} - \text{Co-f. } 3R \times \ddot{M} \\ (\text{Ec. to } n \text{ Lines.}) \end{array} \right\} = \text{Flu. } \frac{x^{m-n}}{r^n - x^n} \text{ ent of } \frac{x^{m-n}}{r^n - x^n}$$

$\mathcal{Q}. E. I.$

COROLLARY.

330. Since, in the Fluent here given, the second Line and the last, the third and the last but one, &c. are respectively equal (*vid. Art. 317.*) the same may also be exhibited, thus;

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} * \text{.....} \text{.....} M \\ \text{Sin. } R \times 2\dot{N} - \text{Co-f. } R \times 2\dot{M} \\ \text{Sin. } 2R \times 2\ddot{N} - \text{Co-f. } 2R \times 2\ddot{M} \end{array} \right\}$$

(Ec. to $\frac{n+1}{2}$ Lines.)

SCHOLIUM.

331. If the Semi-Periphery ABCH of the Circle whose Diameter AH is $2r$, be divided into as many equal

by resolving them into more simple ones.

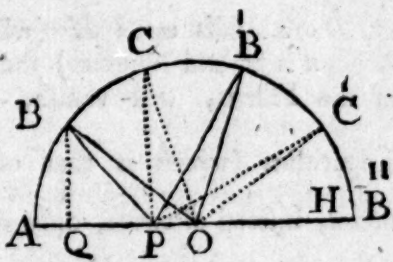
373

equal Parts AB, BC,

CB, BC &c. as there are Units in n (so that

$$AB = \frac{180^\circ}{n} = 2,$$

$$AB = 3 \times \frac{180^\circ}{n} = 6$$



&c. *vid.* Art. 317. and 327.) and in the Radius OA (produced, if necessary) there be taken $OP = x$, and PB, OB &c. be drawn, it will appear (from the said Articles, and from *Prop.* 1.) that the Quantities

$\sqrt{r^2 - 2brx + x^2}$, $\sqrt{r^2 - 2crx + x^2}$ &c. in the former of the two preceding Problems, will here be expounded

by PB, PB &c. respectively: From whence it is also plain, that the Measures N , N &c. of the Angles

whose Sines are $\frac{x \times \text{Sin. } 2}{\sqrt{r^2 - 2brx + x^2}}$, $\frac{x \times \text{Sin. } 6}{\sqrt{r^2 - 2crx + x^2}}$

&c.* will here be expounded by OBP, OBP, &c. &c. * Art. 322. and 323.

Therefore the Fluent of $\frac{x^{m-1}}{r^n + x^n}$, given in the Corollary to the foresaid Proposition, may be thus exhibited,

$$\frac{r^{m-n}}{n} \times \left\{ \begin{array}{l} \text{Sin. } R \times 2 (OBP) - \text{Co-f. } R \times 2 (OA:PB) \\ \text{Sin. } 3R \times 2 (OBP) - \text{Co-f. } R \times 2 (OA:PB) \end{array} \right.$$

&c. &c.

Where the Arch R is $(= m \times \frac{180^\circ}{n}) = m \times AB$, and where $(OA:PB)$ is put (after the Manner of *Cotes*) to express the hyperbolical Logarithm of $\frac{PB}{OA}$. It is also to be observed, that, when the last of the Points B ,

$\overset{I}{B}$, $\overset{II}{B}$ &c. falls upon H (which will always happen when n is an odd Number) the Angle, in the last Line of the Fluent, will vanish, and the corresponding Logarithm (which is that of $\frac{PH}{AO}$) must then be taken, instead of twice, only once.

In the very same Manner it will appear, *that*, the Arcs $\overset{I}{Q}$, $\overset{II}{Q}$ &c. in the second Case, where the Fluent of $\frac{x^{m-1}\dot{x}}{r^n - x^n}$ is sought, will be, respectively, expounded

by AC , AC' &c. also the corresponding Angles N , N' &c. by OCP , OCP' &c. and the Fluent itself by

$$\frac{x^{m-n}}{n} \times \begin{cases} \dots\dots\dots (OA:PC) \\ \text{Sin. } R \times 2 (OCP) - \text{Co-f. } R \times 2 (OA:PC) \\ \text{Sin. } 2R \times 2 (OCP') - \text{Co-f. } 2R \times 2 (OA:PC') \\ \text{&c.} \hspace{10em} \text{&c.} \end{cases}$$

Where the Arch $R (= m \times \frac{360^\circ}{n}) = m \times AC$; and

where, as well as in the preceding Case, all the Arcs, Sines and Co-fines are supposed to have Unity for their Radius.

332. From the Fluents of $\frac{x^{m-1}\dot{x}}{r^n + x^n}$ and $\frac{x^{m-1}\dot{x}}{r^n - x^n}$,

thus given, *those* of $\frac{x^{vn+m-1}\dot{x}}{r^n + x^n}$, $\frac{x^{-vn+m-1}\dot{x}}{r^n + x^n}$,

$\frac{x^{vn+m-1}\dot{x}}{r^n - x^n}$, and $\frac{x^{-vn+m-1}\dot{x}}{r^n - x^n}$, where v denotes any

whole Number, may be very easily deduced; either from *Art.* 283. and 291. or (more readily) by dividing the Numerator by the Denominator, and continuing the Quo-

by resolving them into more simple ones. 375

Quotient to as many Terms as there are Units in v^* . By ^{Art. 150.} which means, if p be put $= vn + m$, $q = vn - m$, and

the Fluents of $\frac{x^{m-1} \dot{x}}{r^n + x^n}$ and $\frac{x^{m-1} \dot{x}}{r^n - x^n}$ be denoted by V

and W respectively, the Fluents, in the four Cases specified above, will be expressed by

$$\frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} + \frac{r^{2n} x^{p-3n}}{p-3n} (v) \pm r^{vn} V,$$

$$\frac{x^{-q}}{-qr^n} - \frac{x^{n-q}}{n-q \cdot r^{2n}} + \frac{x^{2n-q}}{2n-q \cdot r^{3n}} (v) \pm \frac{V}{r^{vn}},$$

$$- \frac{x^{p-n}}{p-n} - \frac{r^n x^{p-2n}}{p-2n} - \frac{r^{2n} x^{p-3n}}{p-3n} (v) + r^{vn} W,$$

$$\text{and, } \frac{x^{-q}}{qr^n} + \frac{x^{n-q}}{n-q \cdot r^{2n}} + \frac{x^{2n-q}}{2n-q \cdot r^{3n}} (v) + \frac{W}{r^{vn}},$$

respectively.

Moreover, from the same Fluents, those of $\frac{z^{\frac{m}{n}q-1} \dot{z}}{e+fz^q}$,

and $\frac{z^{\frac{m}{n}q-1} \dot{z}}{e-fz^q}$ will likewise become known:

For (having transformed the Fluxions here pro-

posed to $\frac{1}{e} \times \frac{z^{\frac{m}{n}q-1} \dot{z}}{1 + \frac{fz^q}{e}}$, &c.) let $\frac{fz^q}{e}$ be put $= x^n$,

or $x = \sqrt[n]{\frac{fz^q}{e}}$; then will $z^{\frac{m}{n}q} = \sqrt[n]{\frac{e}{f}} \times x^m$, and

consequently $\frac{mq}{n} \times z^{\frac{m}{n}q-1} \dot{z} = \sqrt[n]{\frac{e}{f}} \times mx^{m-1} \dot{x}$.

Whence $z^{\frac{m}{n}q-1} \dot{z} = \frac{n}{q} \times \left(\frac{e}{f} \right)^{\frac{m}{n}} \times x^{m-1} \dot{x}$, and $1 \pm$

$\frac{fz^q}{e} = 1 \pm z^n$; and therefore $\frac{z^{\frac{m}{n}q-1} \dot{z}}{e \pm fz^q} \left(= \frac{1}{e} \times \frac{n}{q} \right.$

$\times \left(\frac{e}{f} \right)^{\frac{m}{n}} \times \frac{x^{m-1} \dot{x}}{1 \pm x^n} \Big) = \frac{n}{qe} \times \left(\frac{e}{f} \right)^{\frac{m}{n}} \times \frac{x^{m-1} \dot{x}}{1 \pm x^n}$:

Whose Fluent is given, by *Prob. 4.* or *5.* But, r being here $= 1$, the general Multiplier $\frac{r^{m-n}}{n}$, there gi-

ven, will be barely $= \frac{1}{n}$: Which, drawn into $\frac{n}{qe} \times$

$\left(\frac{e}{f} \right)^{\frac{m}{n}}$, gives $\frac{1}{qe} \times \left(\frac{e}{f} \right)^{\frac{m}{n}}$, for the general Multiplier in this Case.

One thing more, though well known to Mathematicians, it may be proper here to take notice of; and that relates to the Sines and Co-sines of the fore-mention'd Arcs, R , $2R$, $3R$, &c. &c. (multiplying the several Angles and Ratios) some of which Arcs do frequently exceed the whole Periphery: When this happens to be the Case, the Periphery, or 360° , must be subtracted as often as possible, and the Sine and Co-sine of the Remainder be taken. If the Remainder be greater than 180° , the Sine, falling in the lower Semi-Circle, will be negative; if, between 90° and 270° , the Co-sine, falling beyond the Center, will be negative.

PROB. VI.

333. To find the Fluent of $\frac{x^{2m-1} \dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$; where n and m denote any whole positive Numbers, and where the given Expression cannot be resolved into two Binomials (k being less than Unity. Art. 308. and 310.)

2

Let

by resolving them into more simple ones. 377.

Let R be the Arch whose Cosine is k and Radius Unity, and let $\dot{k}$ be the Sine of the same Arch; moreover, let the Arcs $\frac{R}{n}$, $\frac{R+360^\circ}{n}$, $\frac{R+2 \times 360^\circ}{n}$, $\frac{R+3 \times 360^\circ}{n}$ &c. be denoted by $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{Q}'''$, $\mathcal{Q}^{(4)}$

&c. and let $\dot{b}$, $\dot{c}$, $\dot{d}$ &c. and b , c , d &c. express the Sines, and the Co-sines of the same Arcs respectively.

Then will $\frac{nk r^n x^n}{r^{2n} - 2kr^n x^n + x^{2n}} = \frac{\dot{b} r x}{r^2 - 2brx + x^2} + \frac{\dot{c} r x}{r^2 - 2crx + x^2} + \frac{\dot{d} r x}{r^2 - 2drx + x^2}$ &c. (n) by Lemma 4.)
From whence, multiplying the whole Equation by $\frac{x^{m-1} \dot{x}}{nk r^n}$ we have $\frac{x^{n+m-1} \dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}} = \frac{1}{nk r^{n-1}}$ into

$$\frac{\dot{b} x^m \dot{x}}{r^2 - 2brx + x^2} + \frac{\dot{c} x^m \dot{x}}{r^2 - 2crx + x^2} + \frac{\dot{d} x^m \dot{x}}{r^2 - 2drx + x^2} \text{ \&c.}$$

Now, the Fluent of the first Term hereof $\frac{\dot{b} x^m \dot{x}}{r^2 - 2brx + x^2}$
(if M be put for the hyp. Log. of $\frac{\sqrt{x^2 - 2brx + x^2}}{r}$,

and N for the Arch whose Radius is Unity, and Sine

$\frac{x \times \text{Sin. } \mathcal{Q}}{\sqrt{r^2 - 2brx + x^2}} \Bigg)$ will appear (from Prop. 2.) to be =

$$\text{Sin. } \mathcal{Q} \times \frac{x^{m-1}}{m-1} + \text{Sin. } 2\mathcal{Q} \times \frac{r x^{m-2}}{m-2} + \text{Sin. } 3\mathcal{Q} \times$$

$$\frac{r^2 x^{m-3}}{m-3} \text{ \&c. } (m-1) + r^{m-1} \times \text{Sin. } m\mathcal{Q} \times M + \text{Co-sin.}$$

$$\frac{m\mathcal{Q} \times N}{m\mathcal{Q} \times N},$$

From

From whence, if the Arcs whose Sines are

$$\frac{x \times \text{Sin. } \overset{1}{\mathcal{Q}}}{\sqrt{r^2 - 2crx + x^2}}, \frac{x \times \text{Sin. } \overset{2}{\mathcal{Q}}}{\sqrt{r^2 - 2drx + x^2}} \text{ \&c. be repre-}$$

sented by $\overset{1}{M}$, $\overset{2}{M}$ \&c. and the Logarithms whose Num-

$$\text{bers are } \frac{\sqrt{r^2 - 2crx + x^2}}{r}, \frac{\sqrt{r^2 - 2drx + x^2}}{r} \text{ \&c. by}$$

$\overset{1}{N}$, $\overset{2}{N}$ \&c. respectively, the Fluent of the whole Ex-

pression, omitting the general Multiplier $\left(\frac{1}{nkr^{n-1}} \right)$

will be

$$\begin{aligned} & \left[\begin{array}{c} \text{Sin. } \overset{2}{\mathcal{Q}} \\ \text{Sin. } \overset{1}{\mathcal{Q}} \\ \text{Sin. } \overset{2}{\mathcal{Q}} \\ \text{Sin. } \overset{3}{\mathcal{Q}} \\ \text{\&c.} \end{array} \right] \times \frac{x^{m-1}}{m-1} + \left[\begin{array}{c} \text{Sin. } 2\overset{2}{\mathcal{Q}} \\ \text{Sin. } 2\overset{1}{\mathcal{Q}} \\ \text{Sin. } 2\overset{2}{\mathcal{Q}} \\ \text{Sin. } 2\overset{3}{\mathcal{Q}} \\ \text{\&c.} \end{array} \right] \times \frac{rx^{m-2}}{m-2} + \left[\begin{array}{c} \text{Sin. } 3\overset{2}{\mathcal{Q}} \\ \text{Sin. } 3\overset{1}{\mathcal{Q}} \\ \text{Sin. } 3\overset{2}{\mathcal{Q}} \\ \text{Sin. } 3\overset{3}{\mathcal{Q}} \\ \text{\&c.} \end{array} \right] \\ & \times \frac{r^2 x^{m-3}}{m-3} \text{ (\&c. to } m-1 \text{ Terms)} \end{aligned}$$

$$+ r^{m-1} \times \left[\begin{array}{c} \text{Sin. } m\overset{2}{\mathcal{Q}} \times \overset{1}{M} + \text{Co-f. } m\overset{2}{\mathcal{Q}} \times \overset{1}{N} \\ \text{Sin. } m\overset{1}{\mathcal{Q}} \times \overset{2}{M} + \text{Co-f. } m\overset{1}{\mathcal{Q}} \times \overset{2}{N} \\ \text{Sin. } m\overset{2}{\mathcal{Q}} \times \overset{3}{M} + \text{Co-f. } m\overset{2}{\mathcal{Q}} \times \overset{3}{N} \\ \text{Sin. } m\overset{3}{\mathcal{Q}} \times \overset{4}{M} + \text{Co-f. } m\overset{3}{\mathcal{Q}} \times \overset{4}{N} \\ \text{\&c.} \quad \text{\&c.} \end{array} \right]$$

But, the Sines of the first Column being those of an arithmetical Progression (whose common Difference is

$\frac{360^\circ}{n}$) which arises by dividing the whole Periphery into n equal Parts, their Sum will, therefore, be equal to Nothing.

Moreover, the Sines of the second Column, having $\frac{2 \times 360^\circ}{n}$ for the common Difference of their respective

Arcs do, also, divide the whole Periphery (twice taken) into n equal Parts, and therefore destroy each other.

The same is likewise true, with regard to the Sines of every other Column (except the last of all) when $m-1$ is less than n . But, if m be greater than n , the Arcs, in the Column, whose Place from the first, inclusive, is denoted by n , being expressed by n^2 , n^3 , n^4 &c. (or R , $R+360^\circ$, $R+2 \times 360^\circ$ &c.) whereof the common Difference is the whole Periphery; the Sines of that Column do not destroy one another, but each is equal to that of the first Arc R (Vid. Art. 314. and 318.) and consequently their Sum equal to $n \times \text{Sin. } R$.

In like Manner, if m be greater than $2n$, the Series, continued to $m-1$ Terms, will take in the Column,

where the Arcs are $2n^2$, $2n^3$, $2n^4$ &c. (or $2R$, $2R+2 \times 360^\circ$, $2R+4 \times 360^\circ$ &c.) whereof the Sine of each is, also, equal to the Sine of the first ($2R$) and therefore their Sum $= n \times \text{Sin. } 2R$.

Thus, also, it will appear that the Sines of the Column whose Distance from the first, inclusive, is $3n$ (when m is greater than $3n$) will be each equal to $\text{Sin. } 3R$; &c. &c.

Therefore, seeing all the Columns do actually vanish, except those above specified; whose Places from the Beginning are denoted by n , $2n$, $3n$ &c. and whose corresponding Terms, or Multipliers are, therefore,

represented by $\frac{r^{n-1} x^{m-n}}{m-n}$, $\frac{r^{2n-1} x^{m-2n}}{m-2n}$, $\frac{r^{3n-1} x^{m-3n}}{m-3n}$

&c. it is evident that the whole Expression will be reduced to

$$\begin{aligned} & \overline{\text{Sin. } R} \times \frac{nr^{n-1}x^{m-n}}{m-n} + \overline{\text{Sin. } 2R} \times \frac{nr^{2n-1}x^{m-2n}}{m-2n} \\ & + \overline{\text{Sin. } 3R} \times \frac{nr^{3n-1}x^{m-3n}}{m-3n} \text{ \&c.} \\ & + r^{m-1} \text{ into } \left\{ \begin{array}{l} \text{Sin. } mQ \times M + \text{Co-f. } mQ \times N \\ \text{Sin. } mQ' \times M' + \text{Co-f. } mQ' \times N' \\ \text{Sin. } mQ'' \times M'' + \text{Co-f. } mQ'' \times N'' \\ \text{Sin. } mQ''' \times M''' + \text{Co-f. } mQ''' \times N''' \\ \text{\&c.} \qquad \qquad \qquad \text{\&c.} \end{array} \right. \end{aligned}$$

Which, multiply'd by $\frac{1}{nkr^{n-1}}$, the foresaid, general,

$$\begin{aligned} \text{Multiplicator, gives } & \overline{\text{Sin. } R} \times \frac{x^{m-n}}{m-n \cdot k} + \overline{\text{Sin. } 2R} \times \\ & \frac{r^n x^{m-2n}}{m-2n \cdot k} + \overline{\text{Sin. } 3R} \times \frac{r^{2n} x^{m-3n}}{m-3n \cdot k} \text{ \&c.} \\ & + \frac{r^{m-n}}{nk} \times \left\{ \begin{array}{l} \text{Sin. } mQ \times M + \text{Co-f. } mQ \times N \\ \text{Sin. } mQ' \times M' + \text{Co-f. } mQ' \times N' \\ \text{Sin. } mQ'' \times M'' + \text{Co-f. } mQ'' \times N'' \\ \text{Sin. } mQ''' \times M''' + \text{Co-f. } mQ''' \times N''' \\ \text{\&c.} \qquad \qquad \qquad \text{\&c.} \end{array} \right. \end{aligned}$$

for the true Fluent of $\frac{x^{n+m-1}}{r^{2n}-2kr^n x^n+x^{2n}}$: Where
the former Part of the Expression must be continued to
as many Terms as there are Units in $\frac{n-1}{n}$ (the Re-
mainder, if any, being neglected.) Q. E. I.
Co-

COROLLARY.

334. If the Quotient arising from the Division of m by n (when the former exceeds) be denoted by v , and the Remainder by t ; or, which is the same, if $vn+t=m$, it is evident the Arcs m° , m° , m° &c. which are respectively equal to $m^{\circ} + m \times \frac{360^{\circ}}{n}$, $m^{\circ} + 2m \times \frac{360^{\circ}}{n}$, $m^{\circ} + 3m \times \frac{360^{\circ}}{n}$, &c. (by Construction) will also be equal to $m^{\circ} + v \times 360^{\circ} + t \times \frac{360^{\circ}}{n}$, $m^{\circ} + 2v \times 360^{\circ} + 2t \times \frac{360^{\circ}}{n}$ &c. whereof the Sines and Co-sines (omitting $v \times 360^{\circ}$, $2v \times 360^{\circ}$ &c. the Multiples of the whole Periphery) are the same with those of $m^{\circ} + t \times \frac{360^{\circ}}{n}$, $m^{\circ} + 2t \times \frac{360^{\circ}}{n}$ &c. respectively.

Therefore, if the Arcs of the Progression, whereof the first Term is m° , and the common Difference $t \times \frac{360^{\circ}}{n}$, be represented by T , T' , T'' &c. respectively; it

follows that the Fluent of $\frac{x^{n+m-1}}{r^{2n}-2kr^n x^n + x^{2n}}$ (or, $\frac{x^{n+vn+t-1}}{r^{2n}-2kr^n x^n + x^{2n}}$) will, also, be truly expressed by

$$\overline{\text{Sin. } R} \times \frac{x^{m-n}}{m-n \cdot k} + \overline{\text{Sin. } 2R} \times \frac{r^n x^{m-2n}}{m-2n \cdot k} + \overline{\text{Sin. } 3R} \times \frac{r^{2n} x^{m-3n}}{m-3n \cdot k} \text{ \&c. } \left(\frac{m-1}{n} \right)$$

$$+ \frac{r^{m-n}}{nk} \begin{cases} \text{Sin. } T \times M + \text{Co-f. } T \times N \\ \text{Sin. } \dot{T} \times \dot{M} + \text{Co-f. } \dot{T} \times \dot{N} \\ \text{Sin. } \ddot{T} \times \ddot{M} + \text{Co-f. } \ddot{T} \times \ddot{N} \\ \text{Sin. } \dddot{T} \times \dddot{M} + \text{Co-f. } \dddot{T} \times \dddot{N} \\ \text{\textit{Etc.}} \qquad \qquad \text{\textit{Etc.}} \end{cases}$$

In the very same Manner the Fluent of $\frac{x^{n+m-1}}{r^{2n} + 2kr^n x^n + x^{2n}}$ (where the Sign of the second Term is positive) will be exhibited; if R be taken to denote the Arch whose Co-sine is $-k$; which will, in this Case, be greater than a Quadrant.

PROPOSITION VII.

335. To find the Fluent of $\frac{x^{n-m-1} \dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}}$; under the Restrictions mentioned in the last Problem.

Let every thing remain as before: Then we shall

$$\text{have } \frac{x^{n-m-1} \dot{x}}{r^{2n} - 2kr^n x^n + x^{2n}} = \frac{1}{nkr^{n-1}} \text{ into } \frac{\dot{b}x^{-m} \dot{x}}{r^2 - 2brx + x^2}$$

$$+ \frac{\dot{c}x^{-m} \dot{x}}{r^2 - 2crx + x^2} (n) \text{ Whereof the Fluent (by Prob. 3.)}$$

$$\text{appears to be } \frac{1}{nkr^{n-1}} \text{ into}$$

$$- \left\{ \begin{matrix} \text{Sin. } 2 \\ \text{Sin. } \dot{2} \\ \text{Sin. } \ddot{2} \\ \text{\textit{Etc.}} \end{matrix} \right\} \times \frac{x^{1-m}}{m-1 \cdot r^2} - \left\{ \begin{matrix} \text{Sin. } 2 \\ \text{Sin. } \dot{2} \\ \text{Sin. } \ddot{2} \\ \text{\textit{Etc.}} \end{matrix} \right\} \times \frac{x^{2-m}}{m-2 \times r^3}$$

383

$$-\left\{\begin{array}{l} \text{Sin. } 3^{\text{Q}} \\ \text{Sin. } 3^{\text{Q}} \\ \text{Sin. } 3^{\text{III}} \\ \text{E}^{\text{C}}. \end{array}\right\} \times \frac{x^{3-m}}{m-3 \cdot r^4} (m) +$$

$$\frac{1}{r^{m+1}} \times \left\{\begin{array}{l} -\text{Sin. } m^{\text{Q}} \times M + \text{Co-f. } m^{\text{Q}} \times N \\ -\text{Sin. } m^{\text{I}} \times \dot{M} + \text{Co-f. } m^{\text{I}} \times \dot{N} \\ -\text{Sin. } m^{\text{II}} \times \ddot{M} + \text{Co-f. } m^{\text{II}} \times \ddot{N} \\ \text{E}^{\text{C}}. \end{array}\right\}$$

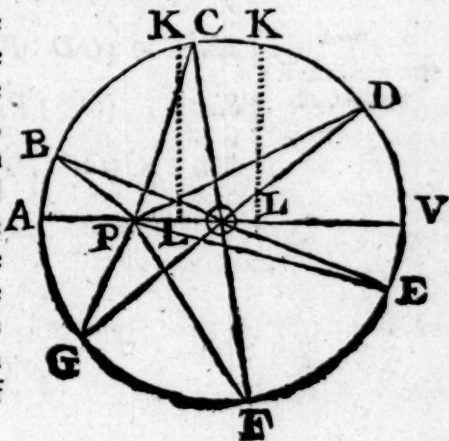
$$-\overline{\text{Sin. } R} \times \frac{x^{n-m}}{m-n \cdot kr^{2n}} - \overline{\text{Sin. } 2R} \times \frac{x^{2n-m}}{m-2n \cdot kr^{3n}}$$

$$\begin{aligned}
& - \overline{\text{Sin. } 3R} \times \frac{x^{3n-m}}{\overline{m-3n} \cdot \dot{k}r^{4n}} \left(\text{to } \frac{m}{n} \text{ Terms} \right) \\
& + \frac{1}{\dot{k}nr^{n+m}} \times \left\{ \begin{aligned} & - \text{Sin. } T \times M + \text{Co-f. } T \times N \\ & - \text{Sin. } \dot{T} \times \dot{M} + \text{Co-f. } \dot{T} \times \dot{N} \\ & - \text{Sin. } \ddot{T} \times \ddot{M} + \text{Co-f. } \ddot{T} \times \ddot{N} \end{aligned} \right. \\
& \qquad \qquad \qquad \mathcal{E}c. \qquad \qquad \qquad \mathcal{E}c.
\end{aligned}$$

Q. E. I.

SCHOLIUM.

336. If, from the Center O, of the Circle ABCD, whose Radius OA, or OV, is r , there be taken OL equal to k and OP= x ; and if the Arch AB be to the Arch AK, whose Co-sine is $\pm k$, as 1 to n ; and each of



384 *Of the Fluents of Rational Fractions,*

of the Arcs BC, CD, DE &c. be taken equal to $\frac{360^\circ}{n}$ &c. &c. Then the Angles R, Q, Q' &c. specified

(in the two preceding Problems) being here expounded by AK, AB, AC &c. respectively, we have

$PB = \sqrt{r^2 - 2brx + x^2}$, $PC = \sqrt{r^2 - 2crx + x^2}$ &c. (Vid. Art. 317. and 323.) Whence, also, the Angles

N, N', N'' &c. whose Sines are $\frac{x \times \text{Sin. Q}}{\sqrt{r^2 - 2brx + x^2}}$,

$\frac{x \times \text{Sin. Q'}}{\sqrt{r^2 - 2crx + x^2}}$, $\frac{x \times \text{Sin. Q''}}{\sqrt{r^2 - 2drx + x^2}}$ &c. will here be equal to B, C, D &c. Therefore the Fluents of

$\frac{x^{n+m-1}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$, and $\frac{x^{n-m-1}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$ (there given) will, also, be truly defined by

$$\frac{x^{m-n}}{m-n} + \frac{\text{Sin. } 2R}{\text{Sin. } R} \times \frac{r^n x^{m-2n}}{m-2n} + \frac{\text{Sin. } 3R}{\text{Sin. } R} \times \frac{r^{2n} x^{m-3n}}{m-3n}$$

(to $\frac{m-1}{n}$ Terms)

$$+ \frac{r^{m-n}}{nx \text{ Sin. } R} \times \left\{ \begin{array}{l} \text{Sin. } T \times (OB : PB) + \text{Co-f. } T \times (B) \\ \text{Sin. } T' \times (OC : PC) + \text{Co-f. } T' \times (C) \\ \text{Sin. } T'' \times (OD : PD) + \text{Co-f. } T'' \times (D) \\ \text{Sin. } T''' \times (OE : PE) + \text{Co-f. } T''' \times (-E) \\ \text{Sin. } T^{(n)} \times (OF : PF) + \text{Co-f. } T^{(n)} \times (-F) \end{array} \right.$$

&c. &c.

And

by resolving them into more simple ones.

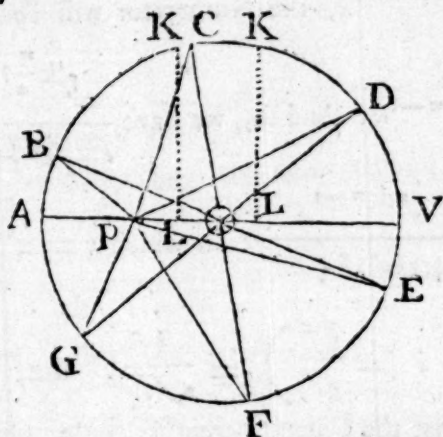
385

And by $-\frac{x^{n-m}}{m-n \cdot r^{2n}} - \frac{\text{Sin. } 2R}{\text{Sin. } R} \times \frac{x^{2n-m}}{m-2n \cdot r^{3n}} -$

$-\frac{\text{Sin. } 3R}{\text{Sin. } R} \times \frac{x^{3n-m}}{m-3n \cdot r^{4n}} \left(\frac{m}{n}\right)$

$+\frac{1}{nr^{n+m} \times \text{Sin. } R} \times \left\{ \begin{array}{l} -\text{Sin. } T \times (OB:PB) + \text{Co-f. } T \times (B) \\ -\text{Sin. } T' \times (OC:PC) + \text{Co-f. } T' \times (C) \\ -\text{Sin. } T'' \times (OD:PD) + \text{Co-f. } T'' \times (D) \\ -\text{Sin. } T''' \times (OE:PE) + \text{Co-f. } T''' \times (-E) \\ -\text{Sin. } T'''' \times (OF:PF) + \text{Co-f. } T'''' \times (-F) \\ \text{\&c.} \end{array} \right\}$

respectively.



Where the Arc AK (or R) will be greater than a Quadrant when the Sign of k is positive ; but less, when

negative ; and where the Arcs T, T', T'' &c. denote an arithmetical Progression, whose first Term (T) is equal to $m \times AB$, and whereof the common Difference is equal to $\frac{360^\circ}{n}$ (or BC) multiply'd by m , when m is less

than n ; but otherwise by the Remainder, of m divided by n .

C c

387.

337. Hence the Fluent of $\frac{z^{q \pm \frac{m}{n} q - 1}}{e \mp fz^q + gz^{2q}}$, where q is any Number, either whole or broken, may be very easily deduced: For, having transform'd the Denominator to $g \times \frac{e}{g} \mp \frac{fz^q}{g} + z^{2q}$, put $\frac{e}{g} = r^{2n}$, $\frac{f}{g} = 2kr^n$, and $z^q = x^n$; and then it will become $= g \times \frac{r^{2n} \mp 2kr^n x^n + x^{2n}}{r^{2n} \mp 2kr^n x^n + x^{2n}}$: Moreover, $z^{q \pm \frac{m}{n} q}$ being $= \sqrt[n]{x^{n \pm m}}$, and $q \pm \frac{m}{n} q \times z^{q \pm \frac{m}{n} q - 1} z = x^{\pm m} \times x^{n \pm m - 1} x$, the Numerator will be reduced to

$$\frac{n}{q} \times x^{n \pm m - 1} x: \text{ And so, we have } \frac{z^{q \pm \frac{m}{n} q - 1}}{e \mp fz^q + gz^{2q}} =$$

$$\frac{n}{qg} \times \frac{x^{n \pm m - 1} x}{r^{2n} \mp 2kr^n x^n + x^{2n}}: \text{ In which } x = z^n, r =$$

$$\sqrt[n]{\frac{e}{g}}, \text{ and } k \left(= \frac{\frac{1}{2}f}{gr^n} \right) = \frac{\frac{1}{2}f}{\sqrt[n]{eg}}. \text{ But, it may be}$$

observ'd, that the Fluent hereof is, only, given when

$\frac{\frac{1}{2}f}{\sqrt[n]{eg}}$ (or its Equal k) is less than Unity *. Therefore,

* Art. 333. if $\frac{1}{2}f$ be greater than $\sqrt[n]{eg}$; or if the Values of e and g are unlike, with regard to positive and negative, so that $\sqrt[n]{eg}$ is impossible, the above Solution fails. But, here, the given Trinomial may be resolv'd into two Binomials (by Art. 310.) and, from thence, the Fluent may be found at two Operations (by Prob. 4. and 5.)

For,

by resolving them into more simple ones. 387

For, by feigning $e \mp fy + gy^2 = 0$, in order to such a Resolution, we get $\frac{\pm \frac{1}{2}f + \sqrt{\frac{1}{4}f^2 - eg}}{g}$, and

$\frac{\pm \frac{1}{2}f - \sqrt{\frac{1}{4}f^2 - eg}}{g}$ for the Roots of that Equation,

or the two first Terms of the required Binomials: Which therefore are always possible when $\frac{1}{4}f^2 - eg$ is positive, or when the foregoing Solution fails.

By denoting the said Roots by H and K , the Trinomial $e \mp fz^q + gz^{2q}$ is resolved into $g \times H - z^q \times K - z^q$,

from whence $\frac{z^{q \pm \frac{m}{n}q-1}}{e \mp fz^q + gz^{2q}}$ is reduced to

$\frac{z^{q \pm \frac{m}{n}q-1}}{g \times K - H \times H - z^q} + \frac{z^{q \pm \frac{m}{n}q-1}}{g \times H - K \times K - z^q}$, whose

Fluent is given by *Art.* 332.

338. By proceeding the same Way the Fluent of

$\frac{z^{\frac{m}{n}q-1}}{e + fz^q + gz^{2q} + bz^{3q}}$ may likewise be found: For,

since one, at least, of the three Roots of the Equation $e + fz + gy^2 + by^3 = 0$, must be possible, the proposed Fluxion, if it cannot be resolved into three Binomials, may, however, be reduced to one Binomial and one Trinomial; and so, be brought under the foregoing Forms: But this being a Speculation too much out of the Way of common Use to be farther pursued, I shall here conclude this Section, with observing, that, when k , in the original Trinomial, above specified, is neither less, nor greater than Unity, the Fluent cannot then be had directly, from either of the preceding Methods; but must be found by Comparison from the Fluent of

$\frac{x^{n \pm m-1}}{r^n \pm x^n}$. *Vid.* *Art.* 289.

SECTION VI.

The Manner of investigating Fluents, when Quantities, and their Logarithms; Arcs and their Sines, &c. are involved together: With other Cases of the like Nature.

PROB. I.

339. **S**UPPOSING $\mathcal{Q}$ and n to denote given Quantities; 'tis propos'd to find the Fluent of $x^n \times \mathcal{Q}^x$.

Let $\mathcal{Q}^x \times Ax^n + Bx^{n-1} + Cx^{n-2} \text{ \&c.}$ be assumed for the Fluent required: Then the Fluxion thereof, which is

$$\begin{aligned} & \text{* Art. 252. } \mathcal{Q}^x \times \text{hyp. Log. } \mathcal{Q}^x \times Ax^n + Bx^{n-1} + Cx^{n-2} \text{ \&c.} + \\ & \mathcal{Q}^x \times nx^{n-1} + \overline{n-1} \cdot Bx^{n-2} + \overline{n-2} \cdot Cx^{n-3} \text{ \&c.} \\ & \text{must consequently be } = x^n \times \mathcal{Q}^x: \text{ And therefore, by} \\ & \text{putting } m \text{ for the hyp. Log. of } \mathcal{Q}, \text{ we have} \\ & \left. \begin{aligned} & mAx^n + mBx^{n-1} + mCx^{n-2} + mDx^{n-3} \text{ \&c.} \\ & -x^n + nAx^{n-1} + \overline{n-1} \cdot Bx^{n-2} + \overline{n-2} \cdot Cx^{n-3} \text{ \&c.} \end{aligned} \right\} \parallel \circ \end{aligned}$$

Whence, comparing the Coefficients of the homologous

Terms, we get $A = \frac{1}{m}$, $B = -\frac{nA}{m} = -\frac{n}{m^2}$, $C =$

$-\frac{\overline{n-1} \cdot B}{m} = \frac{n \cdot \overline{n-1}}{m^3} \text{ \&c. and consequently } \mathcal{Q}^x \times$

$Ax + Bx^{n-1} + Cx^{n-2} + \text{ \&c.} = \frac{\mathcal{Q}^x}{m} \times x^n - \frac{nx^{n-1}}{m} +$

$\frac{n \cdot \overline{n-1} \cdot x^{n-2}}{m^2} - \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \cdot x^{n-3}}{m^3} \text{ \&c. Which}$

Series,

Series, it is plain, will always terminate when n is a whole positive Number. $\mathcal{Q}. E. I.$

340. In the preceding Problem the Coefficients A , B , C , &c. of the assumed Series were taken, in the common Way, as constant Quantities; which, because of the general Multiplier $\mathcal{Q}_x$, was sufficient.

But, in other Cases, where a proper Multiplier, to express the mechanical, or logarithmic, &c. Part of the required Fluent, cannot readily be known, it will be convenient to assume a Series for the *Whole* (independent of any general Multiplier) wherein the Quantities A , B , C , D , &c. must be considered as variable.

PROB. II.

341. To find the Fluent of $z^m x^{n-1} \dot{x}$; z being the Hyperbolic-Logarithm of x ; and m and n any given Numbers.

Let there be assumed $Az^m + Bz^{m-1} + Cz^{m-2} + Dz^{m-3}$ &c. = the Fluent of $z^m x^{n-1} \dot{x}$: Then, in Fluxions, we shall have

$$\begin{aligned} &\dot{A}z^m + \dot{B}z^{m-1} + \dot{C}z^{m-2} + \dot{D}z^{m-3} \text{ \&c.} \\ &+ mAz^{m-1}\dot{z} + \overline{m-1}.Bz^{m-2}\dot{z} + \overline{m-2}.Cz^{m-3}\dot{z} \text{ \&c.} \end{aligned} \left. \vphantom{\begin{aligned} &\dot{A}z^m + \dot{B}z^{m-1} + \dot{C}z^{m-2} + \dot{D}z^{m-3} \text{ \&c.} \\ &+ mAz^{m-1}\dot{z} + \overline{m-1}.Bz^{m-2}\dot{z} + \overline{m-2}.Cz^{m-3}\dot{z} \text{ \&c.} \end{aligned}} \right\} \begin{matrix} \parallel \\ \text{\&A} \\ \text{\&B} \\ \text{\&C} \\ \text{\&D} \\ \text{\&E} \\ \text{\&F} \\ \text{\&G} \\ \text{\&H} \\ \text{\&I} \\ \text{\&K} \\ \text{\&L} \\ \text{\&M} \\ \text{\&N} \\ \text{\&O} \\ \text{\&P} \\ \text{\&Q} \\ \text{\&R} \\ \text{\&S} \\ \text{\&T} \\ \text{\&U} \\ \text{\&V} \\ \text{\&W} \\ \text{\&X} \\ \text{\&Y} \\ \text{\&Z} \end{matrix}$$

But $\dot{z} = \frac{\dot{x}}{x}$; whence, by ordering the Equation, there arises

$$\left. \begin{aligned} &\dot{A} \\ &-x^{n-1}\dot{x} \end{aligned} \right\} \times z^m + \left. \begin{aligned} &\dot{B} \\ &\frac{mA\dot{x}}{x} \end{aligned} \right\} \times z^{m-1} + \left. \begin{aligned} &\dot{C} \\ &\frac{m-1.B\dot{x}}{x} \end{aligned} \right\} z^{m-2} \text{ \&c.} = 0$$

Now, by making the Coefficients of the like Powers of z , equal to Nothing, we have $\dot{A} = x^{n-1}\dot{x}$, $A = \frac{x^n}{n}$; $\dot{B} = \left(-\frac{mA\dot{x}}{x}\right) = -\frac{mx^{n-1}\dot{x}}{n}$, $B = -\frac{mx^n}{n^2}$;

$$\dot{C} \left(= -\frac{\overline{m-1} \cdot B\dot{x}}{x} = \right) \frac{m \cdot \overline{m-1} \cdot x^{n-1} \dot{x}}{n^2} \quad C =$$

$$\frac{m \cdot \overline{m-1} \cdot x^n}{n^3} \text{ \&c. and consequently the Fluent sought}$$

$$= \frac{x^n}{n} \text{ into } z^m - \frac{mz^{m-1}}{n} + \frac{m \cdot \overline{m-1} \cdot z^{m-2}}{n^2} -$$

$$\frac{m \cdot \overline{m-1} \cdot \overline{m-2} \cdot z^{m-3}}{n^3} + \frac{m \cdot \overline{m-1} \cdot \overline{m-2} \cdot \overline{m-3} \cdot z^{m-4}}{n^4}$$

\&c. Which, when m is a whole positive Number, will terminate in $m+1$ Terms. Q. E. I.

PROB. III.

342. To find the Fluent of $z^n y$; z being the Arch of a given Circle, and y the Sine corresponding.

Let there be assumed $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3} =$ Fluent of $z^n y$; then, by taking the Fluxion, we shall have

$$\left. \begin{aligned} \dot{A}z^n + \dot{B}z^{n-1} + \dot{C}z^{n-2} + \dot{D}z^{n-3} \text{ \&c.} \\ -z^n y + nAz^{n-1} \dot{z} + \overline{n-1} \cdot Bz^{n-2} \dot{z} \text{ \&c.} \end{aligned} \right\} = 0$$

Whence, putting $\dot{A} - y = 0$, $\dot{B} + nA\dot{z} = 0$, $\dot{C} + \overline{n-1} \cdot B\dot{z} = 0$, $\dot{D} + \overline{n-2} \cdot C\dot{z} = 0$, \&c. we get $A = y$; $\dot{B} = -ny\dot{z}$, $\dot{C} = -\overline{n-1} \cdot B\dot{z}$ \&c.

But, if a and x be taken to denote the Radius and Co-sine of the Arch z , it will appear, from Art. 142. that $y\dot{z} = -a\dot{x}$ and $x\dot{z} = a\dot{y}$: Therefore $\dot{B} = na\dot{x}$, and $B = nax$; also $\dot{C} \left(= -\overline{n-1} \cdot B\dot{z} \right) = -n \cdot \overline{n-1} \cdot ax\dot{z} = -n \cdot \overline{n-1} \cdot a^2 \dot{y}$, and $C = -n \cdot \overline{n-1} \cdot a^2 y$; likewise $\dot{D} \left(= -\overline{n-2} \cdot C\dot{z} \right) = n \cdot \overline{n-1} \cdot \overline{n-2} \cdot a^2 y\dot{z} = -n \cdot \overline{n-1} \cdot \overline{n-2} \cdot a^3 \dot{x}$, and $D = -n \cdot \overline{n-1} \cdot \overline{n-2} \cdot a^3 x$ \&c.

$$\begin{aligned} & \text{Ec. Ec. and consequently } Az^n + Bz^{n-1} + Cz^{n-2} \text{ Ec.} \\ & = yz^n + naxz^{n-1} - n \cdot \overline{n-1} \cdot a^2yz^{n-2} - n \cdot \overline{n-1} \cdot \overline{n-2} \cdot a^3xz^{n-3} + \text{Ec.} \end{aligned}$$

$$= \left\{ \begin{array}{l} y \times z^n - n \cdot \overline{n-1} \cdot a^2 z^{n-2} + n \cdot \overline{n-1} \cdot \overline{n-2} \cdot a^4 z^{n-4} \text{ Ec.} \\ x \times n a z^{n-1} - n \cdot \overline{n-1} \cdot \overline{n-2} \cdot a^3 z^{n-3} + n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} \cdot a^5 z^{n-5} \text{ Ec.} \end{array} \right.$$

Q. E. I.

In the very same Manner the Fluent of $z^n w$, or $z^n \times -\dot{x}$ (w being the Versed-Sine of the Arch z) will be found $= -xz^n + n y a z^{n-1} + n \cdot \overline{n-1} \cdot x a^2 z^{n-2} - n \cdot \overline{n-1} \cdot \overline{n-2} \cdot y a^3 z^{n-3} - n \cdot \overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} \cdot x a^4 z^{n-4} + \text{Ec.}$

PROB. IV.

343. The Quantities, x, y and z being the same as in the preceding Problem; to find the Fluent of $z^n x^r y^m$.

By assuming $Az^n + Bz^{n-1} + Cz^{n-2} + Dz^{n-3} \&c.$ and proceeding as above, we have $\dot{A} = x^r y^m \dot{y}$, $\dot{B} = -nA\dot{z}$, $\dot{C} = -\overline{n-1} \cdot B\dot{z}$, $\dot{D} = -\overline{n-2} \cdot C\dot{z} \&c.$ or (because $\dot{z} = \frac{ay}{x}$) $\dot{B} = -\frac{naA\dot{y}}{x}$, $\dot{C} = -\frac{\overline{n-1} \cdot aB\dot{y}}{x}$, $\dot{D} = -\frac{\overline{n-2} \cdot aC\dot{y}}{x} \&c.$ Therefore, if the Fluent

of $x^r y^m \dot{y}$ (found from Art. 142. and 291.) be denoted by $\mathcal{Q}$; that of $\frac{\mathcal{Q}\dot{y}}{x}$, by R ; that of $\frac{R\dot{y}}{x}$, by S ; that of $\frac{S\dot{y}}{x}$, by $T \&c.$ it follows that the Fluent of $z^n x^r y^m$ will be truly represented by $\mathcal{Q}z^n - naRz^{n-1} + n \cdot \overline{n-1} \cdot a^2 Sz^{n-2} - n \cdot \overline{n-1} \cdot \overline{n-2} \cdot a^3 Tz^{n-3} \&c.$

COROLLARY

344. Since $\dot{y} = -\frac{x\dot{x}}{y} = \frac{x\dot{z}}{a}$ (Vid. Art. 142.) it fol-

lows that $z^n x^r y^m \dot{y}$ is $= -z^n x^{r+1} y^{m-1} \dot{x} = \frac{z^n x^{r+1} y^m \dot{z}}{a}$.

Therefore the Fluents of these two last Expressions are, also, exhibited in the foregoing Series.

345. As the Values of $\mathcal{Q}, R, S \&c.$ in the preceding Articles, are too complex to be pursued in a general Manner, it may not be amiss to illustrate the Method of proceeding by an Example or two.

Let

Let, then the Fluxion proposed be $\frac{zy^2\dot{y}}{x}$: Where n being $=1$, $m=2$, and $r=-1$, we have $\dot{\mathcal{Q}} = \frac{y^2\dot{y}}{x} = \frac{y^2\dot{y}}{\sqrt{a^2-y^2}}$ (because $\sqrt{a^2-y^2}=x$.) Whence $\mathcal{Q} = -\frac{1}{2}y\sqrt{a^2-y^2} + \frac{1}{2}az = -\frac{1}{2}yx + \frac{1}{2}az$ *, and therefore $\dot{R}$ ($=$ * Art. 279. $\frac{\dot{\mathcal{Q}}}{x}$) $= -\frac{1}{2}\dot{y} + \frac{\frac{1}{2}a\dot{z}}{x} = -\frac{1}{2}\dot{y} + \frac{1}{2}\dot{z}$ (because $\frac{a\dot{y}}{x} = \dot{z}$) and consequently $R = -\frac{1}{4}y^2 + \frac{1}{4}z^2$; and so, $\frac{az-yx}{2} \times z + a \times \frac{yy-zz}{4}$, or $\frac{az^2-2xyz+ay^2}{4}$, is the true Fluent of $\frac{zy^2\dot{y}}{x}$ ($= -zy\dot{x} = \frac{y^2z\dot{z}}{a}$. *) • Art. 344.

Again, let the Fluent of $-p\dot{x} \times \overline{z+y}^2$ (expressing the Content of the Solid generated by the Revolution of the *Cycloid*) be required.

Here, the given Expression, in simple Terms, will become $-pz^2\dot{x} - 2pzy\dot{x} - py^2\dot{x}$: Whereof the Fluent of the first Term $-pz^2\dot{x}$, will be had, by making $n=2$, $m-1=0$, and $r+1=0$ (*Vid. Form. 2. in Corol.*)

Where, we therefore, have $\dot{\mathcal{Q}} = \frac{yy}{x} = -\dot{x}$; whence

$\mathcal{Q} = -x$; also $\dot{R} \left(\frac{\dot{\mathcal{Q}}}{x} \right) = -\dot{y}$, and $R = -y$;

likewise $\dot{S} \left(= \frac{R\dot{y}}{x} \right) = -\frac{yy}{x} = \dot{x}$, $S = x$; and

consequently the Fluent of $-z^2\dot{x}$ ($\mathcal{Q}z^n - naRz^{n-1} + n \cdot n-1 \cdot a^2Sz^{n-2}$ &c.) $= -xz^2 + 2ayz + 2a^2x$:

To which, adding the Fluent $\left(\frac{az^2-2xyz+ay^2}{2} \right)$ of the
second

second Term — $2zy\dot{x}$ (found in the preceding Example) and also that of $-y^2\dot{x}$ (or $-a^2\dot{x} + x^2\dot{x}$, found the common Way) we get, in the Whole, $\frac{1}{2}a - x \times z^2 + 2ay - yx \times z + \frac{1}{2}ay^2 + a^2x + \frac{1}{3}x^3$; which, multiply'd by p , and corrected, gives, p into $\frac{1}{2}a - x \times z^2 + 2ay - yx \times z + \frac{1}{2}ay^2 + a^2x + \frac{1}{3}x^3 - \frac{1}{3}a^4$, for the true Fluent that was to be determined.

PROB. V.

346. Supposing H to denote the Fluent of $\overline{k + lz^n}^r \times z^{vn-1}\dot{z}$; to find the whole Fluent of $H \times \overline{a - bz^n}^m \times z^{qn-1}\dot{z}$, (when $a - bz^n$ becomes equal to Nothing.)

By resolving $\overline{k + lz^n}^r \times z^{vn-1}\dot{z}$ into simple Terms, and taking the Fluent, the ordinary Way, we get $H =$

$$\frac{k^r z^{vn}}{n} \times \frac{1}{v} + \frac{rlz^n}{v+1.k} + \frac{r.r-1.l^2z^{2n}}{2.v+2.k^2} \&c. \text{ Which}$$

Value being substituted above, and p wrote instead of $q+v$, we shall have $H \times \overline{a - bz^n}^m \times z^{qn-1}\dot{z} = \frac{k^r}{n} \times \overline{a - bz^n}^m \times z^{qn-1}\dot{z}$ into $\frac{1}{v} + \frac{rlz^n}{v+1.k} + \frac{r.r-1.l^2z^{2n}}{2.v+2.k^2}$

$$+ \frac{r.r-1.r-2.l^3z^{3n}}{2.3.v+3.k^3} \&c.$$

Let, now, the Fluent of $\overline{a - bz^n}^m \times z^{pn-1}\dot{z}$ (in the proposed Circumstance) be denoted by A , and put $t = p+m+1$; then it follows, from Art. 286. (by writing

$$\frac{1}{v} \text{ for } e, \frac{rl}{v+1.k} \text{ for } f, \&c.) \text{ that } \frac{k^r}{n} \times A \text{ into } \frac{1}{v} +$$

$$\frac{p \cdot r}{t \cdot v + 1} \times \frac{al}{bk} + \frac{p \cdot \overline{p+1} \cdot r \cdot \overline{r-1}}{t \cdot \overline{t+1} \cdot 2 \cdot \overline{v+2}} \times \left(\frac{al}{bk} \right)^2 +$$

$$\frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot r \cdot \overline{r-1} \cdot \overline{r-2}}{t \cdot \overline{t+1} \cdot \overline{t+2} \cdot 2 \cdot 3 \cdot \overline{v+3}} \times \left(\frac{al}{bk} \right)^3 + \&c. \text{ will be}$$

the true Value of the Fluent. Q. E. I.

Note, p and $m+1$ must here be positive Quantities * ; * Art. 286.

and it is also requisite that $\frac{l}{k}$ should be greater than —
 $\frac{b}{a}$; otherwise the Fluent will fail.

Ex. 1. Let $\dot{H} = \overline{1-y^2}^{-\frac{1}{2}} \times y$; and let the whole

Fluent of $H \times \overline{1-y^2}^{-\frac{1}{2}} y$, be demanded.

Then, k being $=1$, $l = -1$, $z = y$, $n = 2$, $r = -\frac{1}{2}$, $v = \frac{1}{2}$; also $a=1$, $b=1$, $m = -\frac{1}{2}$, $q = \frac{1}{2}$;
 $p (=q+v) = 1$, $t (=p+m+1) = \frac{3}{2}$, and A (= the whole Fluent of $\overline{1-y^2}^{-\frac{1}{2}} y$) $= 1$; we shall, by substituting these several Values above, get $1 + \frac{1}{3 \cdot 3} +$

$\frac{1}{5 \cdot 5} + \frac{1}{7 \cdot 7} + \frac{1}{9 \cdot 9} + \frac{1}{11 \cdot 11} \&c. =$ Fluent of $H \times \overline{1-y^2}^{-\frac{1}{2}} \times y$ (or $H\dot{H}$) when $y=1$. Which Fluent being also expressed by $\frac{H^2}{2}$, it follows that $\frac{H^2}{2} = \frac{1}{1} +$

$\frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} \&c.$ Where H is $\frac{1}{4}$ of the Periphery of the Circle whose Radius is Unity.

Ex.

396 *The Manner of making Fluents converge.*

Ex. 2. Let $H = \sqrt{c^2 + z^2}^{-\frac{3}{2}} \times z$; to find the Fluent
of $H \times \sqrt{b^2 - z^2}^{-\frac{1}{2}} \times c^2 z$.

Here, $k=c^2$, $l=1$, $n=2$, $r=-\frac{3}{2}$, $v=\frac{1}{2}$; also
 $a=b^2$, $b=1$, $m=-\frac{1}{2}$, $q=\frac{1}{2}$, $p(q+v)=1$, t
 $(p+m+1)=\frac{3}{2}$, and $A (= \text{whole Fluent of } \sqrt{b^2 - z^2}^{-\frac{1}{2}} \times z)$

$\times z) = b$: Whence, by Substitution, we have c^{-3}

$$b \times 1 - \frac{1}{3} \times \frac{b^2}{c^2} + \frac{1}{5} \times \frac{b^4}{c^4} - \frac{1}{7} \times \frac{b^6}{c^6} \quad \&c. \quad \text{which,}$$

multiply'd by c^2 (the Coefficient of z) gives $\frac{1}{c} \times$

$$b - \frac{b^3}{3c^2} + \frac{b^5}{5c^4} - \frac{b^7}{7c^6} \quad \&c. \quad \text{for the true Fluent in this}$$

Cafe: Where the Series is *that* expressing the Arch of
* Art. 142. the Circle whose Tangent is b and Radius c^* ; and is
therefore equal to $c \times \text{Arch}$, whose Radius is Unity and
Tangent $= \frac{b}{c}$: Whence this last Arch (taken without
the Multiplicator c) is the true Value of the Fluent.

S E C T I O N V I I .

Shewing how Fluents, found by Means of Infinite Series, are made to converge.

347. **I**T is found, in Art. 85. that the Fluent of
 $\sqrt{a + cz^n}^m \times dz^{qn-1}z$, in an Infinite Series,
(making $m+q=s$) is expressed by $\frac{\sqrt{a + cz^n}^{m+1}}{qna} \times$

$$1 - \frac{s+1 \cdot cz^n}{q+1 \cdot a} + \frac{s+1 \cdot s+2 \cdot c^2 z^{2n}}{q+1 \cdot q+2 \cdot a^2} - \text{&c.} \quad \text{Whence}$$

it follows (and is evident by bare Inspection) that the

Fluent of $\overline{a-cy^n}^r \times y^{qn-1} y$ (where the second Term under the Vinculum is negative) will be truly defined by

$$\frac{\overline{a-cy^n}^{r+1}}{qna} \times y^{qn} \text{ into } 1 + \frac{s+1 \cdot cy^n}{q+1 \cdot a} + \frac{s+1 \cdot s+2 \cdot c^2 y^{2n}}{q+1 \cdot q+2 \cdot a^2}$$

+ &c. supposing $s=r+q$.

But, besides the Series here given, and Those, in Art. 83. 84. expressing the same Value, the Fluent of

$\overline{a-cy^n}^r \times y^{qn-1} y$ will, yet, admit of another Form, different from all of them; by means whereof and that above, we shall be enabled to draw out some very useful Conclusions.

$$348. \text{ Put } z^n = \frac{ay^n}{a-cy^n}; \text{ then } y^n = \frac{az^n}{a+cz^n}, \text{ and}$$

$$\text{therefore } ny^{qn-1} y = \frac{na^2 z^{qn-1} \dot{z}}{a+cz^n^2}; \text{ also } a-cy^n = \frac{a^2}{a+cz^n},$$

$$\text{and } y^{qn-1} y (= y^{qn-n} \times y^{n-1} y) = \frac{a^{q+1} z^{qn-1} \dot{z}}{a+cz^n^{q+1}}; \text{ and}$$

$$\text{consequently } \overline{a-cy^n}^r \times y^{qn-1} y = a^{2r+q+1} \times$$

$$\overline{a+cz^n}^{-r-q-1} \times z^{qn-1} \dot{z}: \text{ Which Fluxion, so trans-}$$

form'd, being compared with $\overline{a+cz^n}^m \times dz^{qn-1} \dot{z}$; we

have $m = -r-q-1$, $d = a^{2r+q+1}$, and $s (q+m) = -r-1$; whence, by substituting these Values in the first Series, above given, the Fluent sought will

$$\text{be had} = \frac{\overline{a+cz^n}^{-r-q} \times a^{2r+q} \times z^{qn}}{qn} \times 1 + \frac{rcz^n}{q+1 \cdot a}$$

$$+ \frac{r \cdot r-1 \cdot c^2 z^{2n}}{q+1 \cdot q+2 \cdot a^2} + \frac{r \cdot r-1 \cdot r-2 \cdot c^3 z^{3n}}{q+1 \cdot q+2 \cdot q+3 \cdot a^3} \text{ &c.}$$

398 *The Manner of making Fluents converge.*

Which, by restoring y (or writing $\frac{a^2}{a-cy^n}$ and $\frac{ay^n}{a-cy^n}$ for their Equals $a + cz^n$, and z^n) becomes

$$\frac{\overline{a-cy^n}^r \times y^{qn}}{qn} \times 1 + \frac{r}{q+1} \times \frac{cy^n}{a-cy^n} + \frac{r \cdot r-1}{q+1 \cdot q+2} \times \frac{c^2 y^{2n}}{\overline{a-cy^n}^2} \text{ \&c. the true Fluent, of } \overline{a-cy^n}^r \times y^{qn-1}.$$

349. This Fluent may be otherwise found, independent of *that* above, in the following Manner :

It is evident, by taking the Fluxion of $\frac{\overline{a-cy^n}^r \times y^{qn}}{qn}$ (which Quantity would be the Fluent sought, if $\overline{a-cy^n}^r$ was constant) that $\frac{\overline{a-cy^n}^r \times y^{qn}}{qn}$ is = the Fluent of $\overline{a-cy^n}^r \times y^{qn-1} \dot{y}$ — Fluent of $\frac{rc}{q} \times$

$\overline{a-cy^n}^{r-1} \times y^{qn+n-1} \dot{y}$: This Equation, by transposing the last Term, and writing x in the room of $a-cy^n$ (for the Sake of Brevity) will become $Flu. x^r y^{qn-1} \dot{y} = \frac{x^r y^{qn}}{qn} + \frac{rc}{q} \times Flu. x^{r-1} y^{qn+n-1} \dot{y}$. From the very same

Argument (if, instead of r , we substitute $r-1$, $r-2$ &c. successively ; and, for q . write $q+1$, $q+2$, $q+3$, &c. respectively) we shall, also, have

$$Flu. x^{r-1} y^{qn+n-1} \dot{y} = \frac{x^{r-1} y^{qn+n}}{q+1 \cdot n} + \frac{r-1 \cdot c}{q+1} \times$$

$$Flu. x^{r-2} y^{qn+2n-1} \dot{y} ;$$

$$Flu. x^{r-2} y^{qn+2n-1} \dot{y} = \frac{x^{r-2} y^{qn+2n}}{q+2 \cdot n} + \frac{r-2 \cdot c}{q+2} \times$$

$$Flu. x^{r-3} y^{qn+3n-1} \dot{y} ;$$

&c. &c.

Whence, by substituting these Values, one by one, in that of, $Flu. x^r y^{qn-1} j$, we get

$$\begin{aligned}
 Flu. x^r y^{qn-1} j &= \frac{x^r y^{qn}}{qn} + \frac{rc}{q} \times \frac{x^{r-1} y^{qn+n}}{q+1.n} + \frac{r.r-1.c^2}{q.q+1} \\
 &\times Flu. x^{r-2} y^{qn+2n-1} j = \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q.q+1.n} + \\
 &\frac{r.r-1.c^2}{q.q+1} \times \frac{x^{r-2} y^{qn+2n}}{q+2.n} + \frac{r.r-1.r-2.c^3}{q.q+1.q+2} \times \\
 Flu. x^{r-3} y^{qn+3n-1} j &= \frac{x^r y^{qn}}{qn} + \frac{rcx^{r-1} y^{qn+n}}{q.q+1.n} + \\
 &\frac{r.r-1.c^2 x^{r-2} y^{qn+2n}}{q.q+1.q+2.n} + \frac{r.r-1.r-2.c^3 x^{r-3} y^{qn+3n}}{q.q+1.q+2.q+3.n}
 \end{aligned}$$

&c. Where the Law of Continuation is manifest; and where, by making $\frac{x^r y^{qn}}{qn}$ a general Multiplier, we shall have the very Series above exhibited.

350. From the Equality of the two foregoing Expressions, for the Fluent of $a - cy^n \times y^{qn-1} j$, (or $x^r y^{qn-1} j$) the Business of finding Fluents, by infinite Serieses, will, in many Cases, be very much facilitated.

For, in the first Place, it follows (by dividing both by

$$\frac{a - c^n}{qna} \times y^{qn}, \text{ or } \frac{x^{r+1} y^{qn}}{qna} \quad \text{that the Serieses } 1 +$$

$$\frac{s+1.c^n}{q+1.a} + \frac{s+1.s+2.c^2 y^{2n}}{q+1.q+2.a^2} \text{ \&c. and } \frac{a}{x} \times$$

$$1 + \frac{rcy^n}{q+1.x} + \frac{r.r-1.c^2 y^{2n}}{q+1.q+2.x^2} + \frac{r.r-1.r-2.c^3 y^{3n}}{q+1.q+2.q+3.x^3} +$$

&c. must also be equal to each other, let the several
Quan-

400 *The Manner of making Fluents converge.*

Quantities, therein concerned, be what they will (which may be otherwise proved, independent of Fluxions.) Therefore, if in the room of q and s we write any other Quantities p and t , the Equation will, *still*, hold, and

$$\text{will then become } 1 + \frac{\overline{t+1} \cdot cy^n}{\overline{p+1} \cdot a} + \frac{\overline{t+1} \cdot \overline{t+2} \cdot c^2 y^{2n}}{\overline{p+1} \cdot \overline{p+2} \cdot a^2}$$

$$+ \mathcal{E}c. = \frac{a}{x} \times 1 + \frac{rcy^n}{\overline{p+1} \cdot x} + \frac{r \cdot r-1 \cdot c^2 y^{2n}}{\overline{p+1} \cdot \overline{p+2} \cdot x^2} \mathcal{E}c.$$

(t being $= p+r$.)

$$\text{Moreover, if as many Terms of the first Series } 1 + \frac{\overline{s+1} \cdot cy^n}{\overline{q+1} \cdot a} + \frac{\overline{s+1} \cdot \overline{s+2} \cdot c^2 y^{2n}}{\overline{q+1} \cdot \overline{q+2} \cdot a^2} + \frac{\overline{s+1} \cdot \overline{s+2} \cdot \overline{s+3} \cdot c^3 y^3}{\overline{q+1} \cdot \overline{q+2} \cdot \overline{q+3} \cdot a^3}$$

$\mathcal{E}c.$ be taken as are denoted by any given Number v , and the last of them be represented by $\mathcal{Q}$, it is evident, from the Law of the Series, that the first of the re-

maining Terms will be expressed by $\mathcal{Q} \times \frac{s+v}{q+v} \times \frac{cy^n}{a}$;

the second, of them, by $\mathcal{Q} \times \frac{s+v}{q+v} \times \frac{s+v+1}{q+v+1} \times$

$\frac{c^2 y^{2n}}{a^2} \mathcal{E}c.$ and therefore the Sum of all of them (putting

$q+v=p$ and $s+v (=r+q+v) = t$) will be $= \mathcal{Q} \times$

$$\frac{t}{p} \times \frac{cy^n}{a} + \mathcal{Q} \times \frac{t}{p} \times \frac{t+1}{p+1} \times \frac{c^2 y^{2n}}{a^2} + \mathcal{E}c. =$$

$$\frac{t\mathcal{Q}cy^n}{pa} \times 1 + \frac{\overline{t+1} \cdot cy^n}{\overline{p+1} \cdot a} + \frac{\overline{t+1} \cdot \overline{t+2} \cdot c^2 y^{2n}}{\overline{p+1} \cdot \overline{p+2} \cdot a^2} + \mathcal{E}c.$$

$$= \frac{t\mathcal{Q}cy^n}{px} \times 1 + \frac{rcy^n}{\overline{p+1} \cdot x} + \frac{r \cdot r-1 \cdot c^2 y^{2n}}{\overline{p+1} \cdot \overline{p+2} \cdot x^2} \mathcal{E}c.$$

(by writing the Series found above in the room of its Equal) and consequently the whole Series (including

$$\text{the } v \text{ first Terms) } = 1 + \frac{\overline{s+1} \cdot cy^n}{\overline{q+1} \cdot a} +$$

$$\frac{s+1 \cdot s+2 \cdot c^2 y^{2n}}{q+1 \cdot q+2 \cdot a^2} (v) + \frac{t \mathcal{Q} c y^n}{p x} \times 1 + \frac{r c y^n}{p+1 \cdot x} +$$

$$\frac{r \cdot r-1 \cdot c^2 y^{2n}}{p+1 \cdot p+2 \cdot x^2} + \frac{r \cdot r-1 \cdot r-2 \cdot c^3 y^{3n}}{p+1 \cdot p+2 \cdot p+3 \cdot x^3} + \mathcal{E} c.$$

Which, drawn into the general Multiplicator $\frac{x^{r+1} \times y^{qn}}{qna}$

(*vid. Art. 347.*) will give the Fluent of $\frac{a - c y^n}{x} \times y^{qn-1} y$ (or $x^r y^{qn-1} y$) according to a new Form, compounded out of the two preceding ones; where the second Series (the Value of p being large in respect of r) will always converge much faster than the remaining Part of the first, for which it is substituted: But this will, more fully, appear from what follows hereafter. It will be proper to take notice here that the Fluent of

$\frac{a + c z^n}{x} \times z^{qn-1} z$ (the Fluxion first proposed, where the second Term under the Vinculum is positive) will also be had from hence (by writing z for y , m for r , and $-c$ for c) and is therefore equal to $\frac{x^{m+1} z^{qn}}{qna}$

drawn into the Sum of the two following Serieses,

$$1 - \frac{s+1 \cdot c z^n}{q+1 \cdot a} + \frac{s+1 \cdot s+2 \cdot c^2 z^{2n}}{q+1 \cdot q+2 \cdot a^2} - \frac{s+1 \cdot s+2 \cdot s+3 \cdot c^3 z^{3n}}{q+1 \cdot q+2 \cdot q+3 \cdot a^3} (v)$$

$$- \frac{t \mathcal{Q} c z^n}{p x} \times 1 - \frac{m c z^n}{p+1 \cdot x} + \frac{m \cdot m-1 \cdot c^2 z^{2n}}{p+1 \cdot p+2 \cdot x^2} -$$

$$\frac{m \cdot m-1 \cdot m-2 \cdot c^3 z^{3n}}{p+1 \cdot p+2 \cdot p+3 \cdot x^3} + \mathcal{E} c.$$

Where, $s = m + q$, $p = v + q$, $t = s + v$, $x = a + c z^n$, and $\mathcal{Q}$ = the last Term of the first Series continued to v Terms, v being any whole Number, at pleasure. A few Examples will shew the Use of what is above delivered.

351. Ex. 1. Let $\frac{z}{1+z}$, or $\overline{1+z}^{-1} \cdot z$, be propounded.

Which being compared with $\overline{a+cz^n}^m \times z^{qn-1} \cdot z$, we have $a=1, c=1, n=1, x=1+z, m=-1, qn-1=0$, or $q=1$; whence also $s(m+q)=0, p(v+q)=v+1, s(s+v)=v$, and consequently the Fluent itself (by substituting these several Values in the last general Theorem) = z into $1 - \frac{z}{2} + \frac{z^2}{3} - \frac{z^3}{4} (v) - \frac{vz^2}{v+1 \cdot x}$

$$\times 1 + \frac{z}{v+2 \cdot x} + \frac{2 \cdot z^2}{v+2 \cdot v+3 \cdot x^2} + \frac{2 \cdot 3 \cdot z^3}{v+2 \cdot v+3 \cdot v+4 \cdot x^3}$$

&c. Where (2) the last Term of the first Series being $\pm \frac{z^{v-1}}{v}$, the Multiplier $\left(\frac{vz^2}{v+1 \cdot x} \right)$ to the

Second, will be $= \mp \frac{z^v}{v+1 \cdot x}$; and so the Fluent itself

will be reduced to $z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} (v) \mp \frac{z^{v+1}}{v+1 \cdot x} \times$

$$1 + \frac{z}{v+2 \cdot x} + \frac{2 \cdot z^2}{v+2 \cdot v+3 \cdot x^2} + \text{\&c.} \text{ In which}$$

the Signs $-$ and $+$, before z^{v+1} , obtain alternately, according as v is an odd or even Number. But, to shew the Advantage of expressing the Fluent in this Manner, by two different Serieses, let $z=1$, and let v be taken $= 8$; then the Value of the first Series (continued to 8 Terms) being $= 0,6345238$ &c. and That

of the second Series $= \frac{1}{18} + \frac{A}{20} + \frac{2B}{22} + \frac{3C}{24} + \frac{4D}{26}$

$+ \frac{5E}{28}$ &c. (where A, B, C, D &c. denote the Terms

preceding those where they stand) $= 0,0555555 + 0,0027778 + 0,0002525 + 0,0000316 + 0,0000048 + 0,0000009 + 0,0000002 = 0,0586233$; it is evident that

that the Fluent of $\frac{z}{1+z}$, when z becomes $= 1$, will be $= 0,6345238 + 0,0586233 = 0,6931471$: Which is true to the very last Decimal Place; and would have required, at least, 100000 Terms of the first, or common, Series.

352. Ex. 2. Let the Fluent of $\frac{z}{1+z^2}$ (expressing the Arch whose Radius is 1 and Tangent z) be required.

In this Case we have $a=1$, $c=1$, $n=2$, $x=1+zz$, $m=-1$, $qn-1=0$, or $q=\frac{1}{2}$, $s=-\frac{1}{2}$, $p=v+\frac{1}{2}$,

and the Fluent itself $= z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} (v) \pm$

$$\frac{z^{2v+1}}{2v+1 \cdot x} \times 1 + \frac{2 \cdot z^2}{2v+3 \cdot x} + \frac{2 \cdot 4 \cdot z^4}{2v+3 \cdot 2v+5 \cdot x^2} +$$

$$\frac{2 \cdot 4 \cdot 6 \cdot z^6}{2v+3 \cdot 2v+5 \cdot 2v+7 \cdot x^3} \&c. \text{ Where, if } z \text{ be taken}$$

$= 1$, and $v=6$, we shall have $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$

$$\frac{1}{9} - \frac{1}{11} + \frac{1}{26} \times 1 + \frac{1}{15} + \frac{1}{15} \times \frac{2}{17} + \frac{1}{15} \times \frac{2}{17} \times$$

$$\frac{3}{19} \&c. = 0,785398 = \text{the Fluent of } \frac{z}{1+z^2} \text{ when } z$$

$= 1$ ($= \frac{1}{2}$ of the Periphery of the foresaid Circle) Which Number, brought out by taking, only, 8 Terms of the second Series, is more exact than if 100000

Terms of the common Series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \&c.$

had been used. And, if z be taken $= \sqrt{\frac{1}{3}}$ ($=$

Tangent of 30°) and $v=6$, as before, the same Number of Terms will be sufficient to give the Answer, true to twice the Decimal Places above exhibited.

404 *The Manner of making Fluents converge.*

353. Ex. 3. Let the Fluxion proposed be $\sqrt{e^4 + y^4}^{\frac{1}{2}} \times y$.

Here we have, $a=e^4$, $c=1$, $z=y$, $n=4$, $x=e^4+y^4$, $m=\frac{1}{2}$, $q=\frac{1}{4}$, $s(m+q)=\frac{3}{4}$, $p(v+q)=v+\frac{1}{4}$; $t(s+v)=v+\frac{3}{4}$; and therefore the Fluent sought (by

$$\begin{aligned} \text{Substitution) is } &= \frac{x^{\frac{3}{2}} y}{e^4} \text{ into } 1 - \frac{7y^4}{5e^4} + \frac{7 \cdot 11 y^8}{5 \cdot 9 e^8} - \\ &\frac{7 \cdot 11 \cdot 15 y^{12}}{5 \cdot 9 \cdot 13 e^{12}} (v) - \frac{4v+3 \cdot 2 y^4}{4v+1 \cdot x} \times 1 - \frac{2y^4}{4v+5 \cdot x} \\ &- \frac{2 \cdot 2 y^8}{4v+5 \cdot 4v+9 \cdot x^2} - \frac{2 \cdot 6 \cdot 2 y^{12}}{4v+5 \cdot 4v+9 \cdot 4v+13 \cdot x^3} \end{aligned}$$

&c. In which (as in all other Cases) 2 denotes the last Term of the first Series. This Fluent approximates equally fast with those in the foregoing Examples: And it may be observed farther, that the Fluent will always converge, however great the Value of z is taken, if

both a and c , in the general Fluxion $\sqrt{a+cz^n}^m \times z^{qn-1} z$, are positive Quantities. But, if the second Term under the Vinculum be negative, the Case will be otherwise, when that Term becomes greater than half the First;

since the Powers of $\frac{cz^n}{x}$, in the latter Part of the

Fluent, will then form an increasing Geometrical Progression. It may, therefore, be of use to shew how the Theorem may be varied so as to answer in this Case. In order thereto, if in the Equations $s=r+q$, and $1+$

$$\begin{aligned} &\frac{s+1 \cdot cy^n}{q+1 \cdot a} + \frac{s+1 \cdot s+2 \cdot c^2 y^{2n}}{q+1 \cdot q+2 \cdot a^2} \text{ \&c. } = \frac{a}{x} \times \\ &1 + \frac{rcy^n}{q+1 \cdot x} + \frac{r \cdot r-1 \cdot c^2 y^{2n}}{q+1 \cdot q+2 \cdot x^2} \text{ \&c. (given in Art. 350.)} \end{aligned}$$

you write k for r , and p for q , and multiply by $\frac{x}{a}$,
you

you will have $s = k + p$, and $1 + \frac{kcy^n}{p+1.x} +$

$$\frac{k.k-1.c^2y^{2n}}{p+1.p+2.x^2} \mathcal{E}c. = \frac{x}{a} \times 1 + \frac{s+1.cy^n}{p+1.a} +$$

$$\frac{s+1.s+2.c^2y^{2n}}{p+1.p+2.a^2} \mathcal{E}c.$$

Moreover, if the v first Terms of the above Series $1 + \frac{rcy^n}{q+1.x} + \frac{r.r-1.c^2y^{2n}}{q+1.q+2.x^2} \mathcal{E}c.$ be taken, and the last

of them be denoted by $\mathcal{Q}$, it is plain the first of the remaining Terms will be $= \mathcal{Q} \times \frac{r-v+1}{q+v} \times \frac{cy^n}{x}$,

the second $= \mathcal{Q} \times \frac{r-v+1}{q+v} \times \frac{r-v}{q+v+1} \times \frac{c^2y^{2n}}{x^2} \mathcal{E}c.$

and the Sum of them all (putting $q+v=p$, and $r-v=k$) equal to $\frac{k+1.\mathcal{Q}cy^n}{px} \times 1 + \frac{kcy^n}{p+1.x} +$

$$\frac{k.k-1.c^2y^{2n}}{p+1.p+2.x^2} \mathcal{E}c. = \frac{k+1.\mathcal{Q}cy^n}{px} \times \frac{x}{a} \times 1 + \frac{s+1.cy^n}{p+1.a}$$

$$+ \frac{s+1.s+2.c^2y^{2n}}{p+1.p+2.a^2} \mathcal{E}c. \text{ (by the Equation above) and}$$

consequently the Sum of the whole Series ($1 + \frac{rcy^n}{q+1.x}$

$$\mathcal{E}c.) = 1 + \frac{rcy^n}{q+1.x} + \frac{r.r-1.c^2y^{2n}}{q+1.q+2.x^2} +$$

$$\frac{r.r-1.r-2.c^3y^{3n}}{q+1.q+2.q+3.x^3} (v) + \frac{k+1 \times cy^n \mathcal{Q}}{pa} \times$$

$$1 + \frac{s+1.cy^n}{p+1.a} + \frac{s+1.s+2.c^2y^{2n}}{p+1.p+2.a^2} + \mathcal{E}c. \text{ Which,}$$

406 *The Manner of making Fluents converge.*

multiply'd by $\frac{x^r y^{qn}}{qn}$, gives the Fluent of $\overline{a - cy^n}^r$

* Art. 348, $\times y^{qn-1} \cdot y$ (* or $x^r y^{qn-1} \cdot y$) where $k = r - v$, $p = v$
349.

+ q , $s (=k+p) = r+q$ and $x = a - cy^n$. I shall put down one Example of the Use of this last general Expression; where we will take $y \sqrt{2y - y^2}$, or $\overline{2 - y}^{\frac{1}{2}} \times y^{\frac{1}{2}}$ (being the Fluxion of the Area of the Circle whose Radius is Unity and versed Sine y) In which Case, $a=2$, $c=1$, $n=1$, $r=\frac{1}{2}$, $qn-1=\frac{1}{2}$, or $q=\frac{3}{2}$, $k=-v+\frac{1}{2}$, $p=v+\frac{3}{2}$, $s=2$, $x=2-y$; and therefore the

$$\begin{aligned} \text{Fluent sought} &= \frac{2x^{\frac{1}{2}} y^{\frac{3}{2}}}{3} \text{ into } 1 + \frac{y}{5x} - \frac{y^2}{5 \cdot 7x^2} + \\ &\frac{3y^3}{5 \cdot 7 \cdot 9x^3} - \frac{3y^4}{7 \cdot 9 \cdot 11x^4} + \frac{3y^5}{9 \cdot 11 \cdot 13x^5} \quad (v) \mp \\ &\frac{2v-3 \cdot y^2}{2v+3 \cdot 2} \times 1 + \frac{3y}{2v+5} + \frac{3 \cdot 4y^2}{2v+5 \cdot 2v+7} + \\ &\frac{3 \cdot 4 \cdot 5y^3}{2v+5 \cdot 2v+7 \cdot 2v+9} \quad \&c. \end{aligned}$$

$$\begin{aligned} &= 1, \text{ and } v=5, \text{ will become } = \frac{2}{3} + \frac{A}{5} - \frac{B}{7} + \frac{C}{9} \\ &- \frac{5D}{11} + \frac{7E}{2 \times 13} + \frac{3F}{15} + \frac{4G}{17} + \frac{5H}{19} \quad \&c. = 0,785398 \end{aligned}$$

(where $A, B, C \&c.$ denote the several Terms, respectively, without their Signs.) In bringing out which Conclusion, six Terms of the second Series are required: But if y be taken $=\frac{1}{2}$ the Radius of the foresaid Circle, then four Terms of each Series will be more than sufficient to give the same Number of Decimal Places. And it may likewise be observed, that, although no general Rule can be laid down for assigning the Value of v , so as to answer the best in all Cases, yet the Conclusion will,

will, for the general Part, require the fewest Terms, when the Number of those, taken in each Series, is nearly the same.

354. But, after all, another Theorem or Series, still, seems wanting, to express the Value of the whole Fluent, when the Quantity under the *Vinculum* becomes equal to Nothing (which, in the Resolution of Problems, is, commonly, what is required.) For, it is plain the last, above given, answers no better, *here*, than that preceding it; because (the Divisor (x) being Nothing) the former Part of it fails.

In order, therefore, to determine a proper *Form*, to obtain in this Circumstance, it will be requisite to observe, first of all, from Article 286. that the whole

Fluent of $\frac{1}{a-bz^n} \times z^{pn+qn-1} z$, supposing that of $\frac{1}{a-bz^n} \times z^{pn-1} z$ to be denoted by A , will be truly expressed by $\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{a^v A}{b^v}$: In

which $t=m+p+1$; and where it is requisite that the Values of $m+1$ and p should be positive, otherwise, A being infinite, the Fluent (or Comparifon) fails. Hence,

because the whole Fluent of $\frac{1}{a-bz^n} \times z^{pn-1} z$, (when $a-bz^n = 0$) is found $= \frac{a^{m+1}}{m+1 \times nb}$, by the common

Way *, it follows, by writing this Value in the Room of * Art. 77. A , and expounding p by 1, that the whole Fluent of and 78.

$\frac{1}{a-bz^n} \times z^{pn+qn-1} z$ is rightly expressed by $\frac{1}{m+2} \times \frac{2}{m+3} \times \frac{3}{m+4} (v) \times \frac{a^{m+v+1}}{m+1 \times nb^{v+1}}$, or by $\frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3} (v+1) \times \frac{a^{m+v+1}}{v+1 \times nb^{v+1}}$: Whence

Dd 4

That

408 *The Manner of making Fluents converge.*

That of $\overline{a-bz^n}^m \times z^{rn-1}z$, by substituting r instead of $v+1$, will consequently be equal to $\frac{1}{m+1} \times \frac{2}{m+2}$

$\times \frac{3}{m+3} (r) \times \frac{a^{m+r}}{rnb^r}$. Let this Quantity be denoted

by B ; then, by the same Article, the Fluents of the

several Terms of the Series $1, \frac{bz^n}{a}, \frac{b^2z^{2n}}{a^2}, \frac{b^3z^{3n}}{a^3} \&c.$

drawn into the general Multiplicator $\overline{a-bz^n}^m \times z^{rn-1}z$, will be, respectively, expounded by those of the Series $1,$

$\frac{r}{t}, \frac{r \cdot r+1}{t \cdot t+1}, \frac{r \cdot r+1 \cdot r+2}{t \cdot t+1 \cdot t+2} \&c.$ drawn into B ; t be-

ing $=m+r+1$.

If now the Differences of the Quantities $1, \frac{r}{t},$

$\frac{r \cdot r+1}{t \cdot t+1} \&c.$ be, continually, taken $+$; and for $r-t$ its

Equal $-m-1$ be substituted, the Value of any Term of the Series, whose Distance from the first, exclusive,

is denoted by s , or whose corresponding Term, in the preceding Series, is $\frac{b^sz^{sn}}{a^s}$, will be universally ex-

pressed by $1 - \frac{s \cdot \overline{m+1}}{1 \cdot t} + \frac{s \cdot \overline{s-1} \cdot \overline{m+1} \cdot \overline{m+2}}{1 \cdot 2 \times t \cdot t+1} -$

$\frac{s \cdot \overline{s-1} \cdot \overline{s-2} \cdot \overline{m+1} \cdot \overline{m+2} \cdot \overline{m+3}}{1 \cdot 2 \cdot 3 \times t \cdot t+1 \cdot t+2} + \&c.$ Where,

if s be interpreted by $0, 1, 2, 3 \&c.$ successively, you

will have the Values $1, \frac{r}{t}, \frac{r \cdot r+1}{t \cdot t+1} \&c.$ above exhi-

bited: But, if s be taken as a Fraction, then the Value of such an intermediate Term will be found as will give the

† See my *Mathematical Essays*, p. 94.

the Fluent of $\frac{b^s z^{sn}}{a^s} \times \overline{a-bz^n}^m \times z^{pn-1} z$, in any pro-

posed Circumstance of s ; which Fluent, it is evident,

will therefore be expressed by $B \times 1 - \frac{s \cdot m + 1}{1 \cdot t} +$

$\frac{s \cdot s - 1 \cdot m + 1 \cdot m + 2}{1 \cdot 2 \cdot t \cdot t + 1} \&c.$ or its Equal $\frac{1}{m+1} \times$

$\frac{2}{m+2} \times \frac{3}{m+3} (r) \times \frac{a^{m+r}}{rnb^r}$ into $1 - \frac{s \cdot m + 1}{1 \cdot t} -$

$\frac{s-1 \cdot m+2}{2 \cdot t+1} \times E - \frac{s-2 \cdot m+3}{3 \cdot t+2} \times F - \frac{s-3 \cdot m+4}{4 \cdot t+3} \times$

$G \&c.$ (where $E, F, G \&c.$ denote the Terms im-

mediately preceding those where they stand, under their proper Signs.) Whence, dividing by $\frac{b^s}{a^s}$, we have

$\frac{1}{m+1} \times \frac{2}{m+2} (r) \times \frac{a^{m+r+s}}{rnb^{r+ts}} \times 1 - \frac{s \cdot m + 1}{t} -$

$\frac{s-1 \cdot m+2}{2 \cdot t+1} \times E, \&c.$ for the true Fluent of $\overline{a-bz^n}^m \times$

$\frac{z^{pn+sn-1}}{z}$.

From the last Fluent that of $\overline{a-bz^n}^m \times z^{pn-1} z$

(in which p denotes any positive Fraction, proper or improper) is very readily obtained: For, if the

same (when $a-bz^n=0$) be denoted by A ; then the

Fluent of $\overline{a-bz^n}^m \times z^{pn+vn-1} z$ will (according to the

Article above quoted) be expressed by $\frac{p}{p+m+1} \times$

$\frac{p+1}{p+m+2} \times \frac{p+2}{p+m+3} (v) \times \frac{a^v A}{b^v}$; supposing v any

positive

positive Integer. Therefore, by making $\overline{a-bz^n}^m \times z^{rn+m-1}z = \overline{a-bz^n}^m \times z^{pn+vn-1}z$, or $r+s=p+v$, the corresponding Fluents must, also, be equal; that is,

$$\frac{p}{p+m+1} \times \frac{p+1}{p+m+2} (v) \times \frac{a^v A}{b^v} = \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3} (r) \times \frac{a^{m+p+v}}{rnb^{p+v}} \times 1 - \frac{s \cdot m+1}{t} \text{ \&c. And}$$

consequently A (the whole Fluent of $\overline{a-bz^n}^m \times z^{pn-1}z$) = $\frac{p+m+1}{p} \times \frac{p+m+2}{p+1} \times \frac{p+m+3}{p+2} (v) \times \frac{1}{m+1} \times \frac{2}{m+2} \times \frac{3}{m+3} (r) \times \frac{a^{m+p}}{nrnb^p} \times$ into the Series $1 - \frac{s \cdot m+1}{1 \cdot t} - \frac{s-1 \cdot m+2}{2 \cdot t+1} E - \frac{s-2 \cdot m+3}{3 \cdot t+2} F - \frac{s-3 \cdot m+4}{4 \cdot t+3} G \text{ \&c. where } t=r+m+1 \text{ and } s=p+v-r$;

$v-r$; v and r being any whole positive Numbers at pleasure.

355. An Example, or two, of the Use of this Conclusion, may be proper.

1°. Let the *whole Fluent* of $\overline{1-x^2}^{-\frac{1}{2}}x$ (expressing the Length of $\frac{1}{4}$ of the Periphery of the Circle whose Radius is Unity) be demanded. In which Case, a being $=1$, $b=1$, $m=-\frac{1}{2}$, $n=2$, $p=\frac{1}{2}$, $t=r+\frac{1}{2}=\frac{2r+1}{2}$, and $s=v-r+\frac{1}{2}=\frac{2v-2r+1}{2}$, the Fluent

sought will, therefore, (by substituting these Values) be

$$\text{had} = \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} (v) \times \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} (r) \times$$

The Manner of making Fluents converge. 411

$$\frac{1}{2r} \text{ into } 1 - \frac{1 \cdot 2v - 2r + 1}{2 \cdot 2r + 1} - \frac{3 \cdot 2v - 2r - 1}{4 \cdot 2r + 3} E -$$

$$\frac{5 \cdot 2v - 2r - 3}{6 \cdot 2r + 5} F - \frac{7 \cdot 2v - 2r - 5}{8 \cdot 2r + 7} G \text{ \&c. Which,}$$

by expounding v by 5 and r by 3, will become =

$$2,16719 \text{ \&c. into } 1 - \frac{1 \cdot 5}{2 \cdot 7} - \frac{3 \cdot 3}{4 \cdot 9} E - \frac{5 \cdot 1}{6 \cdot 11} F +$$

$$\frac{7 \cdot 1}{8 \cdot 13} G + \frac{9 \cdot 3}{10 \cdot 15} H + \frac{11 \cdot 5}{12 \cdot 17} I + \text{\&c.} = 1,5708.$$

In the bringing out of which Value, all the Terms above exhibited are requisite: But, of the common Series, $1 +$

$$\frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 7} + \text{\&c. more than 10}$$

times that Number of Terms would be necessary to answer with the same Degree of Exactness.

Ex. 2°. Let the Fluxion proposed be $\frac{dx}{x^{\frac{1}{2}} \sqrt{d^2 - x^2}}$

(whose whole Fluent, when $x=d$, expresses the Time of Descent of a heavy Body in half the Arch of a Semi-circle, whose Radius is d *.)

* Art. 207.

Here, by comparing $\frac{1}{\sqrt{d^2 - x^2}} x^{-\frac{1}{2}}$ with

$\frac{1}{a - bz^n} x^{pn-1}$, we have $a=d^2$, $b=1$, $n=2$, $pn-1 = -\frac{1}{2}$, or $p=\frac{1}{4}$; also $s(p+v-r) = v-r+\frac{1}{4}$, $t(r+m+1) = r+\frac{1}{2}$: Whence, by taking r and v , each, equal to 4, the Fluent, itself, comes out =

$$\frac{3}{1} \times \frac{7}{5} \times \frac{11}{9} \times \frac{15}{13} \text{ into } \frac{2}{1} \times \frac{4}{3} \times \frac{6}{5} \times \frac{8}{7} \times$$

$$\frac{d^{\frac{1}{2}}}{8} \text{ into } 1 - \frac{1 \cdot 1}{4 \cdot 9} + \frac{3 \cdot 3}{8 \cdot 11} E + \frac{7 \cdot 5}{12 \cdot 13} F + \frac{11 \cdot 7}{16 \cdot 15}$$

$$G + \text{\&c.} = \frac{11 \cdot 16 d^{\frac{1}{2}}}{13 \cdot 5} \text{ into } 1 - \frac{1}{36} + \frac{9E}{88} + \text{\&c.}$$

$= 2,6215d^{\frac{1}{2}}$: Which is to $2\sqrt{2d}$, the Time of Descent along the vertical Diameter of the foresaid Circle, as 2,6215 to 2,8284, or as 100 to 108, nearly.

After the same Manner the Fluent will be found in other Cases: But, with regard to the assigning of the Values of r and v , it may be observed, that the Answer will, commonly, be brought out with the least Trouble when v is taken greater by an Unit or two than r ; which last Quantity must be greater or less, according as a greater or less Degree of Exactness is necessary. — From the foregoing Expressions, by varying the Values of v and r , a great Number of Theorems, for the Summation of Serieses, may be deduced. But this being foreign to my present Purpose, I am not at Leisure to pursue it here.

356. Hitherto Regard has been had to Fluxions of the Binomial-Kind: But, from thence, the Fluents of Trinomials may also be found; when these last can be reduced to Binomials (*by Art. 307.*) without introducing new Radical Quantities. — Besides which Method, I shall, here, give another, which will answer where that fails, and is also applicable to *Multinomials*.

In order thereto, let the Fluent of $\overline{a+cz^n}^m \times z^{pn-1} \dot{z}$, be denoted by A ; and let it be required to find, from thence, the Fluent of the Radical *Multinomial*, or Infinite Series, $\overline{a+cx^n+dx^{2n}+ex^{3n}+fx^{4n}\&c.}^m \times x^{pn-1} \dot{x}$.

Make $cz^n = cx^n + dx^{2n} + ex^{3n} + \&c.$ and $y = x^{pn}$; then, x^n being $= y^{\frac{1}{p}}$, if this Value be substituted for x^n , in the first Equation, it will become $cz^n = cy^{\frac{1}{p}} + dy^{\frac{2}{p}} + ey^{\frac{3}{p}} \&c.$ Whence, by reverting the Series, (*by Art.*

Art. 275.) $y (x^{pn})$ is found $= z^{pn} + Rz^{pn+n} + Sz^{pn+2n} + Tz^{pn+3n} + \mathcal{E}c.$

Where $R = -\frac{pd}{c}$, $S = \frac{p \cdot \overline{p+3}}{2} \times \frac{d^2}{c^2} - \frac{pe}{c}$, $T = -\frac{p \cdot \overline{p+4} \cdot \overline{p+5}}{6} \times \frac{d^3}{c^3} + p \cdot \overline{p+4} \times \frac{de}{c^2} - \frac{pf}{c} \mathcal{E}c.$

Moreover, by taking the Fluxion of the Equation thus brought out, and dividing by pn , we have $x^{pn-1} \dot{x}$
 $= z^{pn-1} \dot{z} + \frac{p+1}{p} \times Rz^{pn+n-1} \dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1} \dot{z}$
 $+ \frac{p+3}{p} \times Tz^{pn+3n-1} \dot{z} + \mathcal{E}c.$

Now let this Value, with that of $cx^n + dx^{2n} + ex^{3n} + \mathcal{E}c.$ (given above) be substituted in the proposed

Fluxion, and it will become $\overline{a+cz^n}^m \times z^{pn-1} \dot{z} +$
 $\frac{p+1}{p} \times Rz^{pn+n-1} \dot{z} + \frac{p+2}{p} \times Sz^{pn+2n-1} \dot{z} + \mathcal{E}c.$

Also, let v denote the Place, or Distance, of any Term of this Series from the first, exclusive; then the Term itself, drawn into the general Multiplicator, will

be expressed by $\overline{a+cz^n}^m \times \frac{p+v}{p} \Delta z^{pn+vn-1} \dot{z}$ (Δ

being the corresponding Coefficient $R, S, T, \mathcal{E}c.$) and

the Fluent thereof by $\frac{p+v}{p} \Delta \times \overline{a+cz^n}^{m+1} \times z^{pn} \times$

$$\frac{z^{vn-n}}{s+1 \cdot nc} - \frac{qaz^{vn-2n}}{s+1 \cdot snc^2} + \frac{q \cdot q-1 \cdot a^2 z^{vn-3n}}{s+1 \cdot s \cdot s-1 \cdot nc^3} (v) \pm$$

$$\frac{p}{t} \times \frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{p+v}{p} \times \Delta \times \frac{a^v A}{c^v} (\text{Art. 283.})$$

414 *The Manner of making Fluents converge.*

Where, $q=p+v-1$, $s=m+q$, $t=p+m+1$, and the Sign of the last Term is $+$ or $-$, according as v is an even or odd Number. Now, if in the Fluent thus given, v be expounded by 1, 2, 3, 4 &c. successively, it is evident the Fluent of the whole Expression will, in all Circumstances of z , be obtained. But, if the Coefficient c be negative, so that $a+cz^n$ may (by increasing z) become equal to Nothing; then, in that Circum-

stance, the Fluent of the foresaid general Term $\overline{a+cz^n}^m$
 $\times \frac{p+v}{p} \Delta z^{pn+vn-1} \dot{z}$ (or $\overline{a-bz^n}^m \times \frac{p+v}{p} \Delta$
 $z^{pn+vn-1} \dot{z}$, making $-c=b$) being, barely, $= \frac{p}{t} \times$

• Art. 286. $\frac{p+1}{t+1} \times \frac{p+2}{t+2} (v) \times \frac{p+v}{p} \times \frac{\Delta a^v A}{b^v}$ *, it follows that

the whole Fluent of the given Expression, or its Equal,

$\overline{a-bz^n}^m \times z^{pn-1} \dot{z} + \frac{p+1}{p} R z^{pn+n-1} \dot{z} \&c.$ will be truly

represented by $A \times 1 + \frac{\overline{p+1} \cdot Ra}{tb} + \frac{\overline{p+1} \cdot \overline{p+2} \cdot Sa^2}{t \cdot \overline{t+1} \cdot b^2}$
 $+ \frac{\overline{p+1} \cdot \overline{p+2} \cdot \overline{p+3} \cdot Ta^3}{t \cdot \overline{t+1} \cdot \overline{t+2} \cdot b^3} \&c.$ In which, $R = \frac{pd}{b}$,

$S = \frac{p \cdot \overline{p+3}}{2} \times \frac{d^2}{b^2} + \frac{pe}{b}$, $T = \frac{p \cdot \overline{p+4} \cdot \overline{p+5}}{6} \times$

$\frac{d^3}{b^3} + \frac{p \cdot \overline{p+4}}{1} \times \frac{de}{bb} + \frac{pf}{b}$, &c. and $A =$ the Fluent

$\overline{a-bz^n}^m \times z^{pn-1} \dot{z}$, when $a-lz^n = 0$.

357. Hence, if the Fluxion given be of the Trinomial Kind (then, $e, f, \&c.$ vanishing) the whole Fluent
of

of $\overline{a - bx^n + dx^{2n}}^m \times x^{p^{n-1}} \dot{x}$ (when $a - bx^n + dx^{2n} = 0$) will, by substituting for $R, S, T, \&c.$ be $= A \times$

$$1 + \frac{p \cdot \overline{p+1}}{1 \cdot t} \times \frac{ad}{bb} + \frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot \overline{p+3}}{1 \cdot 2 \cdot t \cdot \overline{t+1}} \times \left(\frac{ad}{bb} \right)^2 +$$

$$\frac{p \cdot \overline{p+1} \cdot \overline{p+2} \cdot \overline{p+3} \cdot \overline{p+4} \cdot \overline{p+5}}{1 \cdot 2 \cdot 3 \cdot t \cdot \overline{t+1} \cdot \overline{t+2}} \times \left(\frac{ad}{bb} \right)^3 + \&c.$$

358. If $m+1$ and p are the Halves of any odd Affir-

mative-Numbers, the *Fluent* of $\overline{a - bz^n}^m \times z^{p^{n-1}} \dot{z}$, when $a - bz^n = 0$, will be equal to

$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot (m + \frac{1}{2}) \times 1 \cdot 3 \cdot 5 \cdot 7 \cdot (p - \frac{1}{2})}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot (m+p)} \times \frac{a^{m+p} G}{nb^p}^* ; * \text{ Art. 293. and 298.}$$

G being the Periphery of the Circle whose Diameter is

Unity. Therefore the *Fluent* of $\overline{a - bx^n + dx^{2n} + ex^{3n}} \&c.$

$\times x^{p^{n-1}} \dot{x}$, or its Equal, $\overline{a - bz^n} \times z^{p^{n-1}} \dot{z} + \frac{p \cdot \overline{p+1}}{p}$

$\times R z^{p^{n+1}-1} \dot{z} \&c.$ is found, in this Case, by multiplying the Expression here given, into the foregoing Series, $1 + \frac{p+1 \cdot Ra}{tb} + \&c.$

359. An Example or two will help to shew the Use of what is above delivered.

First, let the *Fluent* of $\sqrt[3]{\frac{\dot{x}}{a^2 - x^2 - \frac{x^4}{raa}}}$

(when the Divisor becomes equal to Nothing) be required.

Then, by comparing $a^2 - x^2 - \frac{x^4}{raa}$ with the

416 *The Manner of making Fluents converge.*

the general Trinomial $a - bx^n + dx^{2n}$ $\times x^{pn-1} \dot{x}$, it appears that a^2 must be, here, wrote in the room of a , and that n, m, p, b and d , will be interpreted by 2,

$-\frac{1}{2}, \frac{1}{2}, 1$, and $-\frac{1}{raa}$ respectively: Whence we

have $t(p+m+1) = 1$, $\frac{1.3.5(m+\frac{1}{2}) \times 1.3.5(p-\frac{1}{2})}{2.4.6(m+p)}$

$\times \frac{a^{m+p}G}{nb^p} = \frac{G}{2}$, and the Fluent sought $= \frac{G}{2} \times$

$$1 - \frac{1.3}{2.2r} + \frac{1.3.5.7}{2.2.4.4r^2} - \frac{1.3.5.7.9.11}{2.2.4.4.6.6r^3} + \mathcal{E}c.$$

360. The second Example shall be; to find the Fluent expressing the *Apside-Angle* in an Orbit described by means of a centripetal Force varying according to any Power of the Distance.

In which Case the given Fluxion being

$$\pm px$$

(Vid. Art. 242.

$$\times \sqrt{p^2 + \frac{2}{n+1} \times x^2 - p^2 + \frac{2x^{n+3}}{n+1}}$$

where A is supposed the higher Apse, and CA (and consequently Cb) equal to Unity) we shall, by putting

$1 - p^2 = \beta$, $\frac{n+3}{2} = v$, and $1 - x^2 = y$, reduce it to

$$\frac{\frac{1}{2} \sqrt{1-\beta} \times y}{1-y \times \sqrt{\beta y + \frac{1-vy-1-y}{1-v}}} = \frac{1}{2} \sqrt{1-\beta} \times$$

$$\beta - \frac{vy}{2} + \frac{v.v-2}{2.3} y^2 - \frac{v.v-2.v-3}{2.3.4} y^3 + \mathcal{E}c. \Big]^{-\frac{1}{2}}$$

$$\times y^{-\frac{1}{2}} y + y^{\frac{1}{2}} y + y^{\frac{3}{2}} y + y^{\frac{5}{2}} y + \mathcal{E}c. \quad \text{Where the Quantity}$$

The Manner of making Fluents converge. 417

tity under the Radical Sign (now answering to the Form above prescribed) being compared with

$$a - bx^n + dx^{2n} + ex^{3n} \text{ \&c. } \sqrt[m]{}, \text{ we have } m = -\frac{1}{2},$$

$$n = 1, b = \frac{v}{2}, \frac{d}{b} = \frac{v-2}{3}, \frac{e}{b} = -\frac{\overline{v-2} \cdot \overline{v-3}}{3 \cdot 4}$$

\&c. Also the Value of p with regard to the first Term $(y^{-\frac{1}{2}}j)$ will be $= \frac{1}{2}$ (because $pn - 1 = -\frac{1}{2}$) likewise its Value in the second Term $(y^{\frac{1}{2}}j)$ is $= \frac{3}{2}$; in the third $= \frac{5}{2}$ \&c. In the first of these Cases we,

$$\text{therefore, have } t(m+p+1) = 1, R(p \times \frac{d}{b}) =$$

$$\frac{v-2}{6}, S = \frac{\overline{v-2} \cdot \overline{4v-5}}{72}, T = \frac{\overline{v-2} \cdot \overline{16v^2-37v+22}}{16 \times 45}.$$

Whence it follows, that the Fluent of the first Term

$$\left(\beta - \frac{vy}{2} + \frac{v \cdot \overline{v-2}}{2 \cdot 3} \cdot y^2 \text{ \&c. } \right)^{-\frac{1}{2}} \times y^{-\frac{1}{2}}j \text{ when the}$$

Quantity under the Radical Sign becomes equal to Nothing (or the Body arrives at its lower Apse) will be

$$\text{truly expressed by } \frac{G}{\sqrt{\frac{1}{2}v}} \text{ into } 1 + \frac{v-2}{2v} \cdot \beta +$$

$$\frac{5 \cdot \overline{v-2} \cdot \overline{4v-5}}{48v^2} \cdot \beta^2 + \frac{7 \cdot \overline{v-2} \cdot \overline{16v^2-37v+22}}{6 \times 48v^3} \cdot \beta^3$$

$$+ \text{ \&c.}$$

In the same Manner it will appear, that the Fluent of the second Term, in that Circumstance, is =

$$\frac{G}{\sqrt{\frac{1}{2}v}} \times \frac{1}{v} \cdot \beta + \frac{5 \cdot \overline{v-2}}{4v^2} \cdot \beta^2 + \frac{35 \cdot \overline{v-2} \cdot \overline{2v-3}}{48v^3} \cdot \beta^3$$

E e

\&c.

418 *The Manner of making Fluents converge.*

$$\text{Sc. that of the Third} = \frac{G}{\sqrt{\frac{1}{2}v}} \times \frac{3}{2v^2} \cdot \beta^2 +$$

$$\frac{35 \cdot v - 2}{12v^3} \cdot \beta^3 \text{ Sc. that of the Fourth} = \frac{5}{2v^3} \cdot \beta^3$$

Sc. Sc.

Whence, the Fluent of the whole Series, by collecting these several Values together, will come out =

$$\frac{G}{\sqrt{\frac{1}{2}v}} \times 1 + \frac{1}{2}\beta + \frac{20v^2 - 5v + 2}{48v^2} \cdot \beta^2 + \frac{112v^3 - 63v^2 - 42v - 8}{6 \times 48v^3} \cdot \beta^3 + \text{Sc. Which, drawn into}$$

$\frac{1}{2} \times 1 - \frac{1}{2}\beta - \frac{1}{8}\beta^2 - \frac{1}{16}\beta^3 - \text{Sc. (the Value of}$

the general Multiplier $\frac{1}{2} \sqrt{1-\beta}$) gives $\frac{G}{\sqrt{2v}} \times$

$$1 * + \frac{v - 2 \cdot 2v - 1}{48} \cdot \frac{\beta^2}{v^2} + \frac{v - 2 \cdot 2v - 1 \cdot 2v - 1}{72}$$

$\times \frac{\beta^3}{v^3} \text{ Sc. for the true Measure of the Angle required,}$

in Parts of the Radius, or Unity: From whence, by writing 180 instead of G , we shall have the same in Degrees: Which, last of all, by restoring n , becomes

$$\frac{180^\circ}{\sqrt{n+3}} \times 1 * + \frac{n-1 \cdot n+2}{24} \times \left[\frac{\beta}{n+3} \right]^2 + \frac{n-1 \cdot n+2 \cdot n+2}{18} \times \left[\frac{\beta}{n+3} \right]^3 \text{ Sc.}$$

Where n is the Exponent of the Law of the Force, whereby the Orbit is described; and β , the Defect of the Square of the Measure of the Celerity, at the higher *Apse*, below That which the Body ought to have to revolve in a Circle, this last being denoted by Unity.

The

The Manner of making Fluents converge. 419

The same Conclusion may be otherwise derived, by bringing $1-y$, in the transformed Fluxion, under the *Vinculum*; but this Way of going to work, though we have but one Series to manage, will prove rather more troublesome than the foregoing.

It will appear from the two preceding Examples, especially the first of them, that this last Method of finding Fluents is, chiefly, useful when all the Terms of the given Expression, after the two first, in respect of these, are but small. Which is a Circumstance that frequently occurs in the Resolution of physical Problems; such as determining the Effect of the Atmosphere's Resistance upon the Vibration of Pendulums; and the *Inequalities* of the Planets arising from their Action on each other. — In short, wherever the Fluent, or the Quantity it expresses, would belong to the Circle, or some other of the Conic-Sections, were it not for the Interposition of some small perturbing Force (whereby new Terms, small in Comparison of the two first, are introduced) the said Method will be found of very great Service.

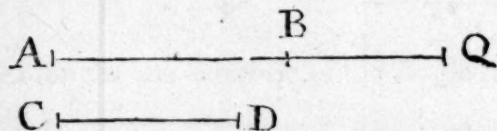
SECTION VIII.

The Use of Fluxions in determining the Motion of Bodies in resisting Mediums.

PROB. I.

361. *Supposing that a Body, let go from a given Point A, with a given Celerity, in a Right-line AQ, is resisted by a Medium (or any Force) acting according to a given Power of the Velocity: To determine the Velocity, and also the Space run over, at the End of a given Time.*

LET the given Celerity at A (measur'd by the Space which would be uniformly described in any proposed Time r) be put $= c$, and that at any other Point B, $= v$; moreover put $AB = x$, and the Time



of its Description $= z$; and let the Resistance, or Force, acting upon the Body at A, be such, that, if the same was to be uniformly continued, the Body would have all its Motion destroyed thereby, in the Time wherein it might move, uniformly, over a given Distance d (CD) with its first Velocity c : Which Time, let be denoted, by t .

Then, since the whole Celerity c would be destroy'd in the Time t , that Part of it which would be uniformly taken away in the Time r , above proposed, will be truly represented by $\frac{r}{t} \times c$; or by $\frac{cc}{d}$; which is equal to it,

because the Spaces (c and d) described with the same Celerity

Celerity are always as the Times (r and t) of their Description; and therefore $\frac{r}{t} = \frac{c}{d}$.

Hence, the Resistance at B being to that at A (by Hypothesis) as v^n to c^n , it follows that the Velocity which might be destroyed in the given Time r , by a Force equal to the Resistance at B, will be expressed by

$$\frac{cc}{d} \times \frac{v^n}{c^n}, \text{ or its Equal } \frac{v^n}{dc^{n-2}} : \text{ Which Expression is,}$$

therefore, the true Measure of the Force of the said Resistance.

Now, it appears, from *Art.* 218. that, if the Force with which the Body is acted on (or the Velocity it would generate in the given Time r) be represented by F , the Relation of the Measures of the Velocity and Space gone over, will be expressed by the Equation $\pm v\dot{v}$

$$= Fx : \text{ From whence, by writing } \frac{v^n}{dc^{n-2}} \text{ instead of}$$

$$F, \text{ we have } -v\dot{v} = \frac{v^n \dot{x}}{dc^{n-2}} \text{ (the Sign of } v\dot{v} \text{ be-}$$

ing negative, because v decreases while x increases *.) * *Art.* 5.

From this Equation, we get $\dot{x} = -dc^{n-2} v^{1-n} \dot{v}$;

whose Fluent is $x = -\frac{dc^{n-2} \times v^{2-n}}{2-n} +$; which,

corrected (by taking $x=0$, and $v=c$) becomes $x =$

$$\frac{-dc^{n-2} \times v^{2-n} + d}{2-n} = \frac{d}{n-2} \times \sqrt[n-2]{\frac{c}{v}} - 1.$$

Moreover, since the Time (z) is to the Time r , as the Distance $\dot{x}$ to the Distance v , we also have $\dot{z} (= \frac{r\dot{x}}{v}) = -rdc^{n-2} v^{-n} \dot{v}$; and consequently $z =$

$\frac{rd}{n-1 \times c} \times \sqrt[n-1]{\frac{c}{v}} - 1 = \frac{t}{n-1} \times \sqrt[n-1]{\frac{c}{v}} - 1$ (by writing t for its Equal $\frac{rd}{c}$): From which Equation

we get $\frac{c}{v} = 1 + \sqrt[n-1]{\frac{z}{t}}$: Likewise,

from the preceding Equation, we get $\frac{c}{v} =$

$1 + \sqrt[n-2]{\frac{x}{d}}$: Which two equal Values being compared together, there, at length, results $x =$

$\frac{d}{n-2}$ into $1 + \sqrt[n-1]{\frac{z}{t}}$ — 1, for the required Relation of x and z . Q. E. I.

COROLLARY.

362. If $n = 2$, or, the Resistance be in the Duplicate Ratio of the Velocity, the Equation exhibiting the Relation of z and v , will be $\frac{c}{v} = 1 + \frac{z}{t}$, or $v =$

$\frac{c}{1 + \frac{z}{t}}$: But the other Equation (the Fluent failing)

becomes impracticable. Here x , the Fluent of —

* Art. 126, $\frac{d\dot{v}}{v}$, will be explicable by $d \times \text{hyp. Log. } \frac{c}{v}$ *, or by $d \times$

$\text{hyp. Log. } 1 + \frac{z}{t}$; because $v = \frac{c}{1 + \frac{z}{t}}$.

In

In the like Manner, when $n=1$, or the Resistance is as the Velocity, the Relation of v , x and z , will be exhibited by the Equations $v = c \times \frac{d-x}{d}$, and $z = t \times$

hyp. Log. $\frac{c}{v} = t \times \text{hyp. Log. } \frac{d}{d-x}$. Which Case, and

that above, are the only two wherein the general Solution fails.

PROB. II.

363. *If a Body, let go from a given Point A with a given Celerity, in a vertical Line CAQ, is acted on by an uniform Gravity, and also by a Medium, resisting according to any given Power of the Velocity; 'tis proposed to determine the Relation of the Times, the Velocities, and the Spaces gone over.*

Let the Notation in the preceding Problem be retained; and let the Force of Gravity, in the given Medium (measured by the Velocity it might generate in the proposed Time r *) be represented by b . Then, * Art. 361. this Value being added to, or subtracted

from $\left(\frac{v^n}{dc^{n-2}}\right)$ the Measure of the Re-

sistance †, according as the Body is in its Ascent, or † Art. 361.

Descent, we thence get $\frac{v^n}{dc^{n-2}} \pm b$ for the whole

Force (F) whereby the Motion, at B , is affected:

Whence (by Art. 218) $\dot{z} (= \frac{-v\dot{v}}{F}) = \frac{-dc^{n-2}v\dot{v}}{v^n \pm bdc^{n-2}}$;

and $\dot{z} (= \frac{r\dot{x}}{v} *) = \frac{-rdc^{n-2}\dot{v}}{v^n \pm bdc^{n-2}}$: Whose Fluents * Art. 361.

may be had, by the Means of circular Arcs, and Logarithms, from Art. 331. Q. E. I.

COROLLARY I.

364. It appears that the Force $\left(\frac{v^n}{dc^{n-2}}\right)$ of the Resistance is to (b) that of Gravity, in the given Medium, as v^n to bdc^{n-2} : Therefore, if this Ratio be expounded by that of v^n to a^n , or a^n be put $= bdc^{n-2}$, it follows that a will express the Celerity with which the Resistance would be equal to the Gravity (since, when $v=a$, the said Ratio becomes that of Equality.) Hence, also, by substituting $\frac{a^n}{b}$ for its Equal dc^{n-2} , we get

$$\dot{x} = \frac{-a^n v \dot{v}}{b \times v^n \pm a^n}, \text{ and } \dot{z} = \frac{-ra^n \dot{v}}{b \times v^n \pm a^n}.$$

COROLLARY II.

365. If the Resistance be in the Duplicate Ratio of the Celerity, our two last Equations will become $\dot{x} = \frac{-a^2 v \dot{v}}{b \times vv \pm aa}$, and $\dot{z} = \frac{-ra^2 \dot{v}}{b \times vv \pm aa}$: From the for-

* Art. 126. mer whereof we get $x = -\frac{a^2}{2b} \times \text{hyp. Log. } \frac{vv \pm aa^*}{cc \pm aa}$
 $= \frac{a^2}{2b} \times \text{hyp. Log. } \frac{cc \pm aa}{vv \pm aa} = \frac{d}{2} \times \text{hyp. Log. } \frac{cc \pm bd}{vv \pm bd}$
 (because, here, $a^2 = bd$.) From whence, when $v=0$, (supposing the Body to ascend) there comes out $x = \frac{d}{2} \times \text{hyp. Log. } 1 + \frac{cc}{aa}$, for the Height (AQ) of the whole Ascent. But, if c be taken $= 0$, or the Body be

be supposed to descend from Rest, we shall then have

$$-\frac{d}{2} \times \text{hyp. Log. } 1 - \frac{vv}{aa} = \text{the Distance } AB \text{ descend-}$$

ed. Whence, if N be put for the Number whose Hyper-

bolical Logarithm is $\frac{2x}{d}$, it follows, (because, $\text{Log. } 1 -$

$$\frac{vv}{aa} = -\frac{2x}{d} = -\text{Log. } N) \text{ that } 1 - \frac{vv}{aa} = \frac{1}{N}, \text{ and}$$

consequently $v = a \sqrt{\frac{N-1}{N}}$. From which, the Di-

stance AB being given, the Velocity acquired in the Fall will be determined. But, if the Body, first, ascends from a given Point A , with a given Celerity c , and the Celerity, acquired in falling, when it arrives, again, at that Point, be required; the same may be exhibited in a more commodious Form, independent of Logarithms,

and will be equal to $\frac{c}{\sqrt{1 + \frac{cc}{aa}}}$; because N , in this

Case, is found above to be $= 1 + \frac{cc}{aa}$. Furthermore,

with regard to the Time (z), we have already found

$$\text{that } z \text{ is } = \frac{-ra^2\dot{v}}{b \times vv + aa}, \text{ or } = \frac{-ra^2\dot{v}}{b \times vv - aa} \left(= \frac{ra^2\dot{v}}{b \times aa - vv} \right) \text{ according as the Motion of the Body}$$

is from, or towards the Center of Force. Therefore

the Time itself, in the former Case, will be $= \frac{ra}{b}$

drawn into the Difference of the two circular Arcs

whose Tangents are $\frac{c}{a}$ and $\frac{v}{a}$, and whereof the com-

mon Radius is Unity*: Whence it follows that the * Art 142.

Time

Time of the whole Ascent will be denoted by $\frac{ra}{b}$ multiply'd into the former of the said Arcs.

But, in the other Case, the Fluent, exhibiting the Time of Descent, is not explicable by the Arcs of a Circle, but by the Difference of the hyperbolical Lo-

Art. 125. garithms of $\frac{a+v}{a-v}$ and $\frac{a+c}{a-c}$ drawn into $\frac{ra}{2b}$ *. There-

fore, when $c = 0$, or the Body falls from Rest, the

Time z will be barely $= \frac{ra}{2b} \times \text{hyp. Log. } \frac{a+v}{a-v} = \frac{ra}{b}$

$\times \text{hyp. Log. } N^{\frac{1}{2}} + \overline{N-1}^{\frac{1}{2}}$ (by substituting the Value of v found above, and ordering the Logarithm as in Art. 303.) This Equation, in the foremention'd Cir-

cumstance, where $N = 1 + \frac{cc}{aa}$, and $v = \frac{c}{\sqrt{1 + \frac{cc}{aa}}}$,

becomes $z = \frac{ra}{b} \times \text{hyp. Log. } \sqrt{1 + \frac{cc}{aa}} + \frac{c}{a}$.

SCHOLIUM.

366. If, according to Sir Isaac Newton, we suppose the Resistance of the Air, to Bodies moving in it, to be in the Duplicate Ratio of the Celerities *; and that

* That the Resistance is as the Square of the Celerity, the Learner may, in some measure, conceive, by considering that the same Body, with a double Velocity, not only puts twice the Number of resisting Particles in Motion, in the same time, but also acts upon each with a double Force; and therefore must suffer a four-fold Resistance, or a Resistance proportional to the Square of the Velocity. This would be strictly true, were it not that the Particles so put in Motion impel others lying before them, and thereby prevent, as it were, the Action of the Body. What Deviation from the foregoing Law may hence arise, is not easy to determine. This, however, seems plain, that the Resistance at the Beginning of any very swift Motion (till the Air in the Way of the Body comes duly to participate of that Motion) will be greater than That sustained by another equal Body, moving with the same Celerity, that has been in Motion some time.

a Ball, in the Time it might move, uniformly, over a Space (d) which is to $\frac{8}{3}$ of its Diameter as the Density of the Ball to that of the Medium, would have all its Motion taken away by a Force equal to that of the Resistance, uniformly continued: Then, from these *Data*, apply'd to the Theorems in the preceding Article, we shall be able to determine the Velocities, and the Times of the perpendicular Ascent and Descent of Bodies near the Earth's Surface; allowing for the Resistance of the Atmosphere.

Thus, for Instance, let a Canon Ball, of four Inches Diameter (whereof the Density, or specific Gravity, is to that of Air as 6000 to 1, nearly) be supposed to be projected, perpendicular to the Horizon, with a Velocity sufficient to cause it to ascend to the Height of half a Mile, or 2640 Feet, *in vacuo*; which Velocity (*by Art. 203.*) will be found to answer to the Rate of about 412 Feet *per Second*: Then, according to the Proportion just now mention'd, it will be as $1 : 6000 :: \frac{8}{3} \times 4 : 64000$ Inches, or 5333 Feet; which is the Value of d in this Case. Therefore, if the Time r , in the preceding Article (which may be assumed at pleasure) be here interpreted by *one Second*, the corresponding Values of d , c and b will be expounded by 5333 F. 412 F. and $32\frac{1}{2}$ F. * respectively. Which Values being substituted in ^{• Art. 202.} the several Equations in the last Article, we shall get

1°. $a (= \sqrt{bd}) = 414$ F. the Velocity, *per Second*, wherewith the Resistance would be equal to the Gravity, or Weight, of the Ball.

2°. $\frac{d}{2} \times \text{hyp. Log. } 1 + \frac{cc}{aa} = 1835$ Feet, the whole Height of the Ascent.

3°. $\frac{ra}{b} \times \text{Arch. whose Tang. is } \frac{c}{a} = 10,08$ Seconds, the whole Time of the Ascent (which is less than the ^s Time, *in vacuo*, by 2,73.)

$$4^{\circ}. v \left(= \frac{c}{\sqrt{1 + \frac{cc}{aa}}} \right) = \overset{F}{292}, \text{ the Velocity, per}$$

Second, acquired in the Descent.

$$5^{\circ}. \text{ Lastly, } \frac{ra}{b} \times \text{hyp. Log. } \sqrt{1 + \frac{cc}{aa} + \frac{c}{a}} =$$

11,30 Seconds, the Time of the Descent.

Note, In this Example the Measure of the absolute Gravity of the Body, *in vacuo*, is taken, instead of its Gravity in Air (the Difference, *there*, being too inconsiderable to be regarded.) But, in Cases where the specific Gravity of the Medium bears a sensible Proportion to that of the Body, the Force of Gravity (*b*) must

be expounded by $32 \frac{1}{2} \times \frac{B-M}{B}$ (instead of $32 \frac{1}{2}$)

where *B* is to *M* as the specific Gravity of the Body to that of the Medium.

P R O B. III.

367. To determine the Resistance, by means whereof a Body, gravitating uniformly in the Direction of parallel Lines, may describe a given Curve.

Let ABC be the given Curve, and let BQ, parallel to the Axis (or any given Line) AH, be the Direction of Gravitation at any Point B: Make PBR perpendicular to AH and BQ; and let AP=*x*, PB=*y*, AB=*z*, BM (Nb) = $\dot{x}$, MN (Bb) = $\dot{y}$, BN = $\dot{z}$, and the Velocity of the Body at B in the Direction PBR = *v*. Then, the Decrease of Velocity in the said Direction,

• Art. 209. which is wholly owing to the Resistance*, being represented by $-\dot{v}$, it follows that the corresponding Decrease of Motion in the Direction BN, arising from the

same Cause, will be expressed by $\frac{\dot{z}}{y} \times -\dot{v} = -\frac{\dot{v}\dot{z}}{y}$;

and, that in the Direction BM, by $-\frac{\dot{v}\dot{x}}{y}$. But, the
of

the said Time: Which being put = to $\left(\frac{v\ddot{x}}{\dot{y}}\right)$ the Value of the same Quantity, given above, we thence have

$v^2 = \frac{\dot{y}\ddot{y}}{\ddot{x}}$: From whence, not only the Velocity, but

the Resistance will be found. But, if you would have the Resistance expressed independent of v ; then let the

Fluxion $(2v\dot{v} = -\frac{\dot{y}^2\ddot{x}}{\ddot{x}\ddot{x}})$ of the last Equation be di-

vided by the Fluent, which will give $\frac{\dot{v}}{v} = -\frac{\frac{1}{2}\ddot{x}}{\ddot{x}}$:

And then, by substituting this Value in $-\frac{\dot{v}\ddot{x}}{v\ddot{x}}$, you will

get $\frac{\ddot{x}\ddot{x}}{2\ddot{x}\ddot{x}}$, for the true Force of the Resistance, *that* of

Gravity (or the Weight of the Body) being expounded by Unity.

The same otherwise.

Let BO be the Radius of Curvature at B, and let OQ be parallel to PB, meeting BM, produced, in Q: Then, if the absolute Gravity, acting in the Direction BQ, be denoted by Unity, its Force in the Direction BO, whereby the Body is retained in the Curve, will

be represented by $\frac{BQ}{BO}$. Therefore, since the Veloci-

ties in Circles are known to be in the Subduplicate Ratio
* Art. 212. of the Radii and of the Forces conjunctly *, the Ve-

locity at B will be rightly expressed by $\sqrt{BO \times \frac{BQ}{BO}}$,

or its Equal $\sqrt{BQ}$. (For the Curve at, and indefinitely near, B may be taken as an Arch of a Circle whose Radius is BO: And it is evident that the Resistance has nothing to do in forcing the Body from the
Tangent,

Tangent, but only serves to retard its Motion so, that it may, every where, bear a due Proportion to the given Force of Gravity acting in the Direction BO.) Hence, putting $BQ = s$, the Increase of the Celerity

in the Time $\left(\frac{\ddot{z}}{\sqrt{s}}\right)$ of describing BN, will be ex-

pressed by the Fluxion of $\sqrt{s}$, or $\frac{\dot{s}}{2\sqrt{s}}$. Moreover,

the Celerity that might be generated by Gravity in the

said Time $\frac{\ddot{z}}{\sqrt{s}}$ being measured thereby, the Increase,

in BN, arising from the same Cause, will therefore be

$= \frac{\ddot{z}}{\sqrt{s}} \times \frac{\dot{s}}{2\sqrt{s}} = \frac{\dot{s}\ddot{z}}{2s}$: Which, being taken from

$\left(\frac{\dot{s}}{2\sqrt{s}}\right)$ the whole Increase, found above, the Re-

mainder, $\frac{\dot{s} - 2\ddot{z}}{2\sqrt{s}}$, will be the Effect of the Resistance:

Which is to the Effect, $\frac{\ddot{z}}{\sqrt{s}}$, of the absolute Gravity

as $\frac{\dot{s} - 2\ddot{z}}{2\ddot{z}}$ to 1. Therefore the Resistance is to the

Gravity (or Weight of the Body) as $\frac{2\ddot{z} - \dot{s}}{2\ddot{z}}$ to U-

nity: Where the Signs are changed, because the two Forces act in contrary Directions.

Because $BO = \frac{\dot{z}^3}{y\ddot{x}}$ *, therefore $s (BO \times \frac{\dot{y}}{\ddot{z}}) =$ * Art. 63.

$\frac{\dot{z}^2}{\ddot{x}} = \frac{y^2 + \dot{x}^2}{\ddot{x}}$ (= the Square of the Celerity) whence

$\dot{s} = \frac{2\dot{x}\ddot{x}\ddot{x} - y^2 + \dot{x}^2 \times \ddot{x}}{\ddot{x}\ddot{x}}$, and consequently the Re-

fistance

Resistance $\frac{2\dot{x} - \dot{s}}{2\dot{x}} = \frac{\dot{y}^2 + \dot{x}^2}{2\dot{x}\ddot{x}} \times \dot{x} = \frac{\dot{x}\ddot{x}}{2\dot{x}\ddot{x}}$, the very same as before.

COROLLARY.

368. If the Resistance be supposed as any given Power of the Velocity drawn into (D) the Density of the Medium; then, from hence, the Density of the Medium, at every Point of the Curve, may be determined: For, the absolute Celerity at B being represented by $\frac{v\dot{z}}{j}$, the Resistance at that Point will, according

to the said Hypothesis, be as $\left(\frac{v\dot{z}}{j}\right)^n \times D$; and therefore the Velocity that would be destroyed thereby, in the Time $\left(\frac{j}{v}\right)$ of describing BN , as $\left(\frac{v\dot{z}}{j}\right)^n \times \frac{Dj}{v}$: Which being put $= \left(-\frac{\dot{v}\dot{z}}{j}\right)$ the Effect of the same Re-

sistance, found above, we thence get $D = \frac{-\dot{v}\dot{y}^{n-2}}{v\dot{z}^{n-1}}$:

Which, by substituting for v and $\dot{v}$, becomes $D =$

$$\frac{\ddot{x}}{2\dot{x}^{n-1} \times \dot{x}^2 - \frac{1}{2}n}$$

In this Corollary, and what, elsewhere, relates to unequal Densities, the Gravity of the Body in the Medium is supposed to continue, every where, the same, or, that the Attraction increases with the Density, so that the Difference between the specific Gravities of the Body and Medium may, at every Point, be a constant Quantity.

E X-

EXAMPLE I.

369. Let the proposed Curve ABC be the common Parabola :

Then, x being here $= \frac{y^2}{a}$, we have $\dot{x} = \frac{2y\dot{y}}{a}$, $\ddot{x} =$

$\frac{2\dot{y}\dot{y}}{a}$ and $\ddot{x} = 0$; and therefore $\frac{\ddot{x}\dot{x}}{2\dot{x}\ddot{x}}^*$ is also $= 0$: * Art. 367;

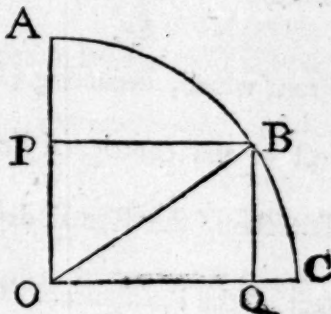
Whence it appears that a Body, to describe this Curve, must move in Spaces intirely void of Resistance.

EXAMPLE II.

376. Let the Curve ABC be taken as a Quadrant of a Circle, whose Radius BO is $= a$.

In this Case we have s (BQ) $= a - x$ ($= AO - AP$) whence $\dot{s} = -\dot{x}$,

and therefore $\frac{2\dot{x} - \dot{s}}{2\dot{x}} =$



* Art. 367;

$\frac{3\dot{x}}{2\dot{x}} = \frac{3PB}{2AO}$ *. From which it is evident, that the * Art. 144;

Velocity is, every where, as $\sqrt{BQ}$, and the Resistance to the Gravity (or Weight of the Body) as $3PB$ to $2OB$.

PROB. IV.

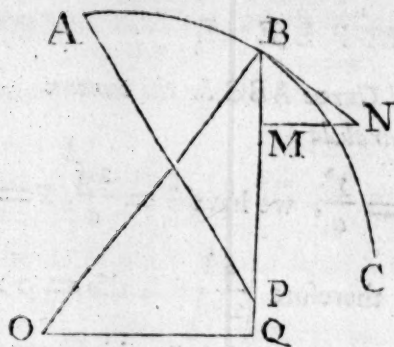
371. The Centripetal Force (F) being given; to find the Resistance and Velocity whereby a Body may describe a given Spiral (or any other, possible, Curve) about the Center of Force.

Let P be the Center of Force, and BO the Radius of Curvature at any Point B in the proposed Curve.

F f

and

* Art. 9.



and let OQ be perpendicular to BPQ; also let BP = y , BQ = s , AB = z , BM = $-j^*$, and BN = $\dot{z}$. Then, it is evident from Art. 367. that the Velocity at B will be expressed by

$$\sqrt{BO \times \frac{BQ}{BO} \times F};$$

or, its Equal, $\sqrt{sF}$. And therefore its Increase in the

Time $\left(\frac{\dot{z}}{\sqrt{sF}}\right)$ of describing BN will be $\frac{s\dot{F} + F\dot{s}}{2\sqrt{sF}}$:

From which, deducting $(F \times \frac{\dot{z}}{\sqrt{sF}} \times \frac{-j}{\dot{z}})$ the Effect of the centripetal Force, in the same Time and

Direction, the Remainder, $\frac{s\dot{F} + F\dot{s} + 2Fj}{2\sqrt{sF}}$, is the Effect of the Resistance. Therefore the Resistance is to

the centripetal Force as $\frac{s\dot{F} + F\dot{s} + 2Fj}{2\sqrt{sF}}$ to $\frac{F\dot{z}}{\sqrt{sF}}$, or

as $\frac{s\dot{F} + F\dot{s} + 2Fj}{2F\dot{z}}$ to Unity.

Q. E. I.

EXAMPLE.

372. Let the Measure (F) of the centripetal Force

be expounded by any Power y^n of the Distance; and

let the Curve be the logarithmic Spiral; putting the

† Art. 61. Co-sine of the given Angle PBN † (to the Radius r)

‡ Art. 74. = c . Then, s being here = y ‡, and $\dot{F} = ny^{n-1}\dot{y}$,

we

we have
$$\frac{\dot{F} + F\dot{z} + 2F\dot{y}}{2F\dot{z}} = \frac{ny''\dot{y} + y''\dot{y} + 2y''\dot{y}}{2y''\dot{z}} = \frac{n+3}{2}$$

$$\times \frac{\dot{y}}{\dot{z}} = \frac{n+3}{2} \times \frac{c}{r}.$$

Hence it appears that the Velocity must be, every
 where, as $y^{\frac{n+1}{2}}$; and the Resistance, to the centripetal
 Force, as $\frac{n+3}{2} \times \frac{c}{r}$ to Unity. But, when $n = -3$,
 $\frac{n+3}{2} \times \frac{c}{r}$ becomes $= 0$; therefore the Body, in this
 Case, must move in Spaces intirely void of Resistance;
 agreeable to *Art.* 233. And, if $n+3$ be negative, an
 accelerating, instead of a resisting Force, will be required.

SCHOLIUM.

373. If the Density of a Medium, wherein a Body
 moves, be either uniform, or varies according to a
 given Law, the Nature of the Curve, or Trajectory,
 may be determined from what is delivered in the pre-
 ceding Pages.

Thus, for Example, let the Density be supposed every
 where the same, and the Resistance as the Square of the

Celerity; then, from *Art.* 368. we have $\frac{\ddot{x}}{x\dot{x}} = D$;

which, in order to exterminate $\dot{x}$, may be transformed

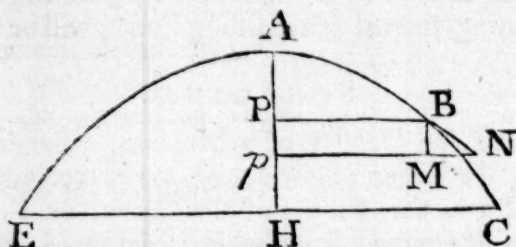
to $\ddot{x}\dot{x} = \overline{y\dot{y}} + \overline{x\dot{x}} \times D^2\dot{x}\dot{x}$: Where, D being a constant
 Quantity (depending upon the given Density of the
 Medium) the Value of x will be found, as is taught in

Sect. 2. *Art.* 268. 271. and comes out $= \frac{y^2}{p} + \frac{Dy^3}{3p}$

$+ \frac{D^2y^4}{12p}$ &c. In which p is put to denote the Para-

meter of the Curve at the Vertex, or highest Point A, (to be determin'd from the Force of Gravity and the given Velocity of the Body at that Point.) This Solution answers near enough when the Resistance is but small in Proportion to the Gravity; in other Circumstances, the Series not converging, it becomes useless: For which Reason, and because the Case above specified is That supposed to obtain, in respect to the Air near the Earth's Surface, and its Resistance to Bodies moving therein, I shall shew, by a different Method, how the Nature of the Curve may be investigated.

In order thereto, let the Celerity at the highest Point, A, above the Plane of the Horizon EC, be denoted by c ; and let a be the Celerity with which the Resistance is equal to the Gravity (*vid. Art. 365. and 366.*)



Moreover, let d be put for the Distance over which the Ball might uniformly move in the Time that the Medium would destroy all its Motion, was the Resistance to continue the same, all along, as at the first Instant (Which Distance, according to Sir *Isaac Newton*, is, always, in Proportion to $\frac{8}{3}$ of the Ball's Diameter, as the Density of the Ball to that of the Medium.)

Then it will be, as $d : z$ (BN) $:: \frac{vz^2}{j}$, the absolute

Celerity at B, to $\left(\frac{vz^2}{dy}\right)$ the Part thereof that would

be uniformly destroyed by the Resistance in the Time of describing BN, with the Velocity at B: Which

Value being also expressed by $\frac{-vz}{j}$ (*vid. Art. 367.*) we

there-

therefore have $\frac{v\dot{z}^2}{dy} = -\frac{\dot{v}\dot{z}}{y}$; whence $\frac{\dot{z}}{d} = \frac{-\dot{v}}{v}$, and

consequently, by taking the Fluent, $\frac{z}{d} = -\text{hyp.}$

Log. v ; which corrected (by putting $z=0$, and $v=c$) gives $\frac{z}{d} (= \text{hyp. Log. } c - \text{hyp. Log. } v) = \text{hyp. Log. } \frac{c}{v}$.

Furthermore, since (by Hypothesis) the Resistance with the Celerity $\frac{v\dot{z}}{y}$ (at B) is to the Force of Gravity,

or the Resistance with the Celerity a , as $\frac{vv\dot{z}\dot{z}}{y\dot{y}}$ to a^2 ;

and it appears, from the aforesaid Article, that the same

Ratio is also universally expressed by that of $\frac{-\dot{v}\dot{z}}{v\dot{x}}$ to

1, it follows, from the Equality of these Ratios, that

$\frac{\dot{z}\ddot{x}}{y\dot{y}}$ is $= -\frac{a^2\dot{v}}{v^3}$. But, in order to the Resolution of

the Equation thus given, let the Tangent of the Angle PBA (or N) which the Ordinate, PB, makes with the Curve (supposing Radius Unity) be, every where,

represented by w : Then, because $\dot{x}=w\dot{y}$, $\dot{z}(\sqrt{y^2+x^2})$

$= y\sqrt{1+w^2}$, and $\ddot{x}=\dot{w}y$ (y being constant) we

shall, by substituting these Values in the foresaid Equ-

ation, get $-\frac{a^2\dot{v}}{v^3} = \dot{w}\sqrt{1+w^2}$; whereof the

Fluent will be given, $\frac{\frac{1}{2}a^2}{v^2} = \frac{1}{2}w\sqrt{1+w^2} + \frac{1}{2}\text{hyp.}$

Log. $w + \sqrt{1+w^2}$ †: Which corrected (by taking † Art. 126,

$v=c$ and $w=0$) becomes $\frac{\frac{1}{2}a^2}{v^2} - \frac{\frac{1}{2}a^2}{c^2} = \frac{1}{2}w\sqrt{1+w^2}$

+ $\frac{1}{2}\text{hyp. Log. } w + \sqrt{1+w^2}$. But, to shorten the

remaining Part of the Process, let the latter Part of the Equation, or the Fluent of $\dot{w} \sqrt{1+w^2}$ be denoted by $\mathcal{Q}$; then $\frac{aa}{2vv}$ being $= \frac{aa}{2cc} + \mathcal{Q}$, we have $v =$

$$\frac{ac}{\sqrt{aa + 2cc\mathcal{Q}}}; \text{ and consequently } \frac{z}{d} (= \text{hyp. Log. } \frac{c}{v}) \\ = \text{hyp. Log. } \frac{\sqrt{aa + 2cc\mathcal{Q}}}{a} = \frac{1}{2} \text{ hyp. Log. } 1 + \frac{2cc\mathcal{Q}}{aa}.$$

From which two Equations, the Velocity of the Ball, and the Distance it has moved, when its Direction makes any given Angle with the Horizon, may be computed, let the Medium be as dense as it will: Also, from hence, if the Celerity answering to any one given Angle of Direction be known, the Celerity corresponding to any other given Direction may be found, together with the Distance described between the two Positions. For v (in the Descent of the Body) being, *universally*,

equal to $\frac{ac}{\sqrt{aa + 2c^2\mathcal{Q}}}$, the Value of c , expressing

the Celerity at the Vertex A , will be had from that

Equation, and comes out $= \frac{av}{\sqrt{aa - 2v^2\mathcal{Q}}}$; whence

also $z (= d \times \text{hyp. Log. } \frac{c}{v}) = d \times \text{hyp. Log.}$

$$\frac{a}{\sqrt{aa - 2v^2\mathcal{Q}}} = -\frac{1}{2} d \times \text{hyp. Log. } 1 - \frac{2vv\mathcal{Q}}{aa}.$$

From which, the Celerity at A being known, the rest is obvious.

But, in the ascending Part of the Curve EA , both z and $\mathcal{Q}$ must be consider'd as negative, or wrote with contrary Signs: And then, from the foregoing Equations,

$$\text{we shall also get } v = \frac{ac}{\sqrt{aa - 2cc\mathcal{Q}}}, c = \frac{av}{\sqrt{aa + 2vv\mathcal{Q}}},$$

and

$$\text{and } -z = \frac{1}{2} d \times \text{hyp. Log. } \sqrt{1 - \frac{2cc\mathcal{Q}}{aa}} = -\frac{1}{2} d \times \\ \text{hyp. Log. } \sqrt{1 + \frac{2vv\mathcal{Q}}{aa}}; \text{ and, consequently, } z = -\frac{1}{2} d \\ \times \text{hyp. Log. } \sqrt{1 - \frac{2cc\mathcal{Q}}{aa}} = \frac{1}{2} d \times \text{hyp. Log. } \sqrt{1 + \frac{2vv\mathcal{Q}}{aa}} \\ = d \times \text{hyp. Log. } \frac{v}{c} : \text{ Answering in this Case.}$$

It still remains to take some notice of the Values of x and y (in order to have the Form, as well as the Length, of the Curve.) These, indeed, are not so easy to bring out as That of z , given above; nor can they be exhibited in a general Manner, either by circular Arcs, or Logarithms (that I have been able to discover) but may, however, be approximated to any required Degree of Exactness, as will appear from what follows.

Since z ($= AB$) is found $= \frac{1}{2} d \times \text{hyp. Log. } \frac{aa + zc^2\mathcal{Q}}{aa}$, by taking the Fluxion thereof, we get $\dot{z} =$

$$\frac{ccd\mathcal{Q}}{aa + 2cc\mathcal{Q}} = \frac{c^2 d \dot{w} \sqrt{1+w^2}}{aa + 2c^2\mathcal{Q}} \text{ (because } \mathcal{Q} = w\sqrt{1+w^2} \text{)}$$

$$\text{Therefore } y \left(= \frac{\dot{z}}{\sqrt{1+w^2}} \right) = \frac{c^2 d \dot{w}}{aa + 2c^2\mathcal{Q}}; \text{ and } \dot{x}$$

$$(= w\dot{y}) = \frac{c^2 d w \dot{w}}{aa + 2c^2\mathcal{Q}} : \text{ Which Equations, by taking}$$

r to 1, as a^2 to c^2 (or as the Square of the Force of Gravity to the Square of the Resistance at A) are re-

$$\text{duced to } \dot{y} = \frac{d\dot{w}}{r + 2\mathcal{Q}}, \text{ and } \dot{x} = \frac{dw\dot{w}}{r + 2\mathcal{Q}} : \text{ Whence}$$

$$\text{we get } y = d \text{ into } \frac{w}{r + 2\mathcal{Q}} + \frac{\frac{2}{3} \times \sqrt{1+ww^2} - \frac{2}{3}}{r + 2\mathcal{Q}^2} +$$

Ff 4

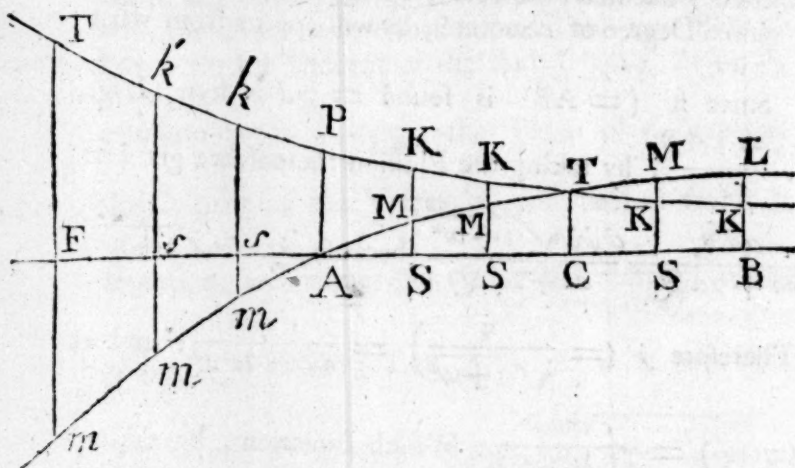
$$\frac{\frac{2}{3}w \times \sqrt{1 + \frac{2}{3}w^2 + \frac{1}{3}w^4} - \frac{2}{3}\mathcal{Q}}{r + 2\mathcal{Q}^3} \&c. \text{ And } x = d \text{ into}$$

$$\frac{\frac{1}{2}w^2}{r + 2\mathcal{Q}} + \frac{\frac{1}{4}w \times \sqrt{1 + ww}}{r + 2\mathcal{Q}^2} - \frac{1}{4}\mathcal{Q} +$$

$$\frac{\frac{1}{8} \times \sqrt{1 + ww}^3 - \frac{1}{8} - \frac{1}{2}\mathcal{Q}^2}{r + 2\mathcal{Q}^3} \&c. \text{ These Expressions}$$

(brought out by assuming $\frac{A}{r + 2\mathcal{Q}} + \frac{B}{r + 2\mathcal{Q}^2} + \&c.$

for the Fluent sought, and proceeding as in *Art.* 340.) converge very fast when r is large in comparison to $\mathcal{Q}$; but in other Cases the required Values will be had, with less Trouble, from the following Method,



Let PKTK and AMTM be two Curves, whereof the Ordinates SK and SM, to the common *Abscissa* w

(= AS) are expressed by $\frac{1}{r + 2\mathcal{Q}}$ and $\frac{w}{r + 2\mathcal{Q}}$ respec-

tively: Then it is plain, from the foregoing Equations, that the Measures of the Areas of the said Curves, multi-

ply'd by d , will truly exhibit the Values of y and x ;
an-

answering to any given Value of w (or AS) the Tangent of the Angle of Direction; or, to speak more geometrically, a Square upon AC (supposing AC = Radius = Unity) will be to either of the said Areas ASKP, or ASM as the given Distance d , to the Value of y or x required.—But now as to a Way for computing these Areas (without which what has been said about them would be to very little Purpose) the Method of *Equi-distant Ordinates* may here be apply'd to very good Advantage (when the foregoing Serieses do not converge) By means whereof the required Quantities may, with a little Trouble, be brought out to a sufficient Degree of Exactness, let the Resistance be as great as it will.

According to the same Way of proceeding, the Values of x and y , in the Ascent of the Ball, will also be found, if the Ordinates sk and sm , generating the re-

quired Areas, be taken, every where, equal to $\frac{1}{r-2\mathcal{Q}}$ and $\frac{w}{r-2\mathcal{Q}}$ (instead of $\frac{1}{r+2\mathcal{Q}}$ and $\frac{w}{r+2\mathcal{Q}}$).

From what has been thus far delivered, it will not be very difficult to calculate (according to the foregoing Hypothesis) all the principal Requisites concerning the Motion and Track of a Ball in the Air, projected with a given Velocity, at a given Elevation; as will be more clearly seen by the Example subjoin'd.

Suppose a Cannon Ball of 4 Inches Diameter (whereof the Weight is nearly 9 Pounds) to be discharged at an Elevation of 45 Degrees, with a Velocity sufficient to carry it to the Distance of one Mile, on the Plane of the Horizon, were it not for the Resistance of the Air.—Then that Velocity, being the same as might be freely acquir'd in a perpendicular Descent of half a Mile*, * Art. 366. will be found to answer to the Rate of 412 Feet, *per Second*, according to *Art. 202.* and 366. From whence it is also plain, that the Distance d (so often mention'd above) will here be expounded by 5333 Feet; and that the Celerity (a) with which the Resistance would be equal to the Gravity (or Weight of the Ball) answers to the Rate of about 414 Feet *per Second*.

More-

Moreover, since the Tangent of the Angle of Elevation, or the first Value of w , is given equal to Unity (or Radius) we have $\mathcal{Q} (\frac{1}{2} w \sqrt{w^2 + 1} + \frac{1}{2} \text{hyp. Log. } w + \sqrt{w^2 + 1}) = 1.1478$: From which, and $v (=$

$412 \sqrt{\frac{1}{2}})$, we get $z (= \frac{1}{2} d \times \text{hyp. Log. } 1 + \frac{2vv\mathcal{Q}}{aa})$

$= 2025$ Feet $=$ the Arch described in the whole As-

cent. Also $c (= \frac{v}{\sqrt{1 + \frac{2vv\mathcal{Q}}{aa}}}) = 199 \frac{1}{3}$ Feet, for

the Rate of the Velocity, per Second, at the highest

Point: Whence $r (= \frac{aa}{cc}) = 4,314$; by Means

whereof the greatest Altitude of the Ball, and the horizontal Distance corresponding thereto will likewise be found: For let AF, in the preceding Figure, be taken $= 1$ (the given Value of w) and let the same be divided into three Parts by equi-distant Ordinates (which Number will answer sufficiently exact) then the successive Values of w , for the Ordinates AP, ks , ks and TF, being $0, \frac{1}{3}, \frac{2}{3}$ and 1 , those of $\mathcal{Q}$ will be $0, 0.3394, 0.713$, and 1.1478 , and the Ordinates themselves (or the cor-

responding Values of $\frac{1}{r - 2\mathcal{Q}} =$ to $0.2318, 0.2751,$

0.3463 and 0.4953 , respectively. From whence, by adding the two Extremes to three times the Sum of the two middle Terms, and dividing the whole by 8, we get 0.3239 for the Value of a mean Ordinate $\dagger$: Which, as AF is here equal to Unity, is also the Measure of the required Area AFTP: Which, therefore, being multiply'd by $5333 (d)$ gives 1727 Feet, for the horizontal Distance made good in the whole Ascent. In

$\dagger$ See p. 117. of my *Mathematical Dissertations*.

the same Way the Area AFm is found $= 0.1828$. Whence the greatest Height of the Ball appears to be $(= 0.1828 \times 5333) = 975$ Feet.

By taking $AC=1$, and repeating the Operation (only changing $r-2\mathcal{Q}$ to $r+2\mathcal{Q}$) the Area $ACTP$ will come out $= 0.1883$, and $ATC = 0.0875$; which multiply'd by 5333 (as above) give 1004 F. and 467 F. for the Amplitude, and the Distance descended, from the highest Point, when the Direction of the Ball makes an Angle with the Horizon equal to that in which it was projected.

But, to have the Direction when the Ball strikes the Ground, and the whole Amplitude of the Projection, we must find the Value of the Tangent AB , when the Area ABL is equal to (0.1828) the Area AFm (so that the Descent, from the highest Point, may become equal to the whole Ascent.) In order thereto, let 0.0875 (ATC) be deducted from 0.1828 (AFm) and the Remainder 0.0953 will be $= CTBL$; this, divided by TC (0.1513) quotes 0.63 ; which would be the Value of CB , if all the Ordinates CT , SM , &c. were equal: But, as it is obvious from the Nature of the Problem, and from the Law of the Ordinates already computed, that BL will be something greater than CT , and consequently CB less than 0.63 —I therefore suppose the Value of CB may be about 0.56 ; and, accordingly, proceed to compute the Area of $CBLT$ answering to this Number; by means of CT (0.1513) and BL (0.1852) and one intermediate Ordinate SM (0.1715) and find

it (from the Approximation $\frac{CT+BL+4SM}{6} \times CB$)

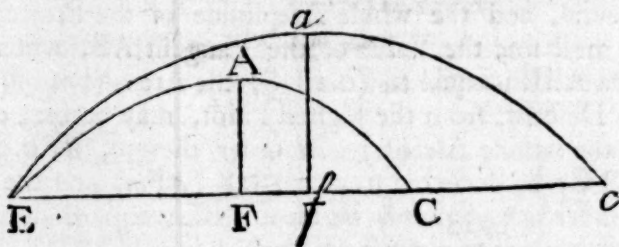
to come out $= 0.0955$: Which is so near the required Value 0.0953 , that it will be altogether needless to repeat the Operation. It is evident from hence, that the Tangent (AB) of the Angle of Direction, when the Ball strikes the Ground, is 1.56 ; answering to $57^{\circ} 26'$: From whence, $CBKT$ being found $= 0.0752$, the whole Area $ABKP$ will be had $= 0.2635$, and consequently $0.2635 \times 5333 = 1405$ F. = the Amplitude in the whole Descent.

Furthermore, from the said Value of w and that of c ($= 199\frac{1}{3}$) given above, we get z ($= \frac{1}{2} d \times \text{hyp.}$

$\text{Log. } 1 + \frac{2cc^2}{aa}$) $= 1788$ Feet, for the Arch described

in the Descent; and also $v = 142\frac{1}{2}$ F. which multiply'd by 1.8527, the Secant of $57^\circ : 20'$, gives 264 F. for the Celerity of the Ball, *per Second*, at the End of its Flight.

Now, by collecting the principal of the foregoing Conclusions, it appears,



1°. That the Velocity at the highest Point A of the Trajectory will be at the Rate of $199\frac{1}{3}$ Feet, *per Second*: Which is to the Velocity at the highest Point a of the Parabola (Eac) that would be described, were it not for the Resistance, as 2 to 3, nearly.

2°. $EA=2025$ and $Ea=3030$	} Feet.
3°. $EF=1727$ and $Ef=2640$	
4°. $AF=975$ and $af=1320$	
5°. $AC=1788$ and $ac=3030$	
6°. $FC=1405$ and $fc=2640$	

7°. Angle $C=57^\circ : 20'$ and c ($=E$) $=45^\circ$.

8°. Velocity at C to that at E, as 264 to 412, or as 2 to 3, nearly.

These Proportions, between the Distances, in Air and *in vacuo*, hold at an Elevation of 45° , when the Resistance, at going off, is nearly equal to the Gravity, or Weight, of the Ball. If the Velocity be greater than that above specified, or the Body, projected, be,
either,

either, less, or less dense, the Curve will differ, *still*, more from a Parabola.

Hence it evidently appears, that the Effect of the Air's Resistance upon very swift Motions, is too considerable to be intirely disregarded in the Art of Gunnery.—'Tis true the Method given above is, by much, too intricate for common Practice; but when the Law of the Resistance to very swift Motions is once sufficiently established (which, according to some late Experiments, seems to be in a Ratio greater than that of the Square of the Celerity) it will be no very difficult Matter to find out proper Approximations to correct the Proportions in common Use.

SECTION IX.

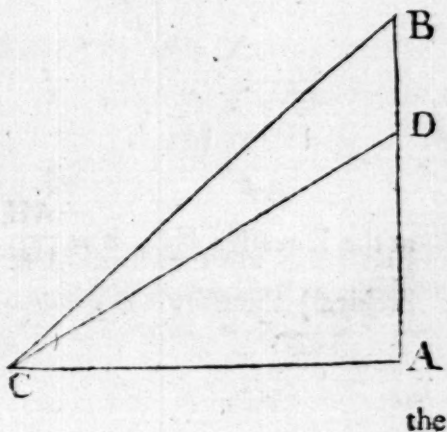
The Use of Fluxions in determining the Attraction of Bodies under different Forms.

PROB. I.

374. **S**upposing AC perpendicular to AB, and that a Corpuscle at C is attracted towards every Point or Particle of the Line AB, by Forces in the reciprocal duplicate Ratio of the Distances; to determine the Ratio of the whole Force whereby the Corpuscle is urged in the Direction CA.

Put $AC=a$, and let AD (considered as variable by the Motion of D towards B) be denoted by x : Then, the Force of a Particle at D being as

$\frac{1}{CD^2}$ (by Hypothesis) its Efficacy in



the

the proposed Direction AC (will by the Resolution of

Forces) be as $\frac{AC}{CD} \times \frac{1}{CD^2} = \frac{AC}{CD^3} = \frac{a}{a^2 + x^2}^{\frac{3}{2}}$: There-

fore $\frac{a\dot{x}}{a^2 + x^2}^{\frac{3}{2}}$ is the Fluxion of the whole Force;

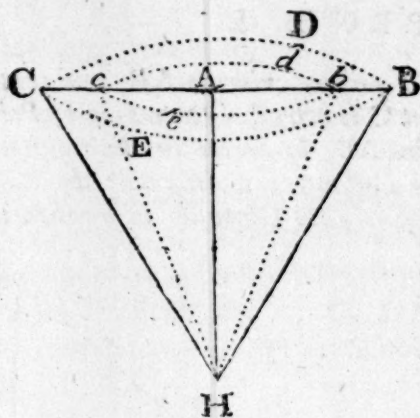
whose Fluent, which (by Art. 85.) is $= \frac{x}{a \times a^2 + x^2}^{\frac{1}{2}}$

$= \frac{AD}{CA \times CD}$, will, when $AD=AB$, be as the Force itself.

Q. E. I.

PROB. II.

375. Supposing BCDE to represent a circular Plane, and that a Corpuscle H, in the Axis thereof AH, is attracted by every Point or Particle of the Plane by Forces in the reciprocal duplicate Ratio of the Distances; to find the whole Force by which the Corpuscle is urged towards the Plane.



Let $AH=a$, and $Hb=x$; then $Ab^2 = x^2 - a^2$; which multiply'd by ($p=3,14159$ &c.) the Area of the Circle whose Radius is U-nity, gives $p \times x^2 - a^2$ for the Area of the Circle $Acdb$: whose Fluxion is $= 2px\dot{x}$. But the Force of a single Particle at b ,

in the Direction HA, is as $\frac{AH}{Hb^3}$, or $\frac{a}{x^3}$ (see the last Problem) therefore the Fluxion of the whole Force is truly

truly defined by $2px\dot{x} \times \frac{a}{x^3}$ or its Equal $\frac{2p\dot{x}}{x^2}$, and the

Force itself by the Fluent of $\frac{2pa\dot{x}}{x^2}$; which (properly

corrected) is $-\frac{2pa}{x} + \frac{2pa}{a} = 2p \times 1 - \frac{a}{x} = 2p \times$

$1 - \frac{AH}{BH}$, when $x = HB$. Q. E. I.

376. In the preceding Problems, we have supposed the Attraction of each Particle, to be as the Square of the Distance inversely; that being the Law which is found to obtain in Nature: But if the Force, according to any other Law of Attraction, be required, the Processes will be very little different.

Thus, let the Attraction be as any Power (n) of the Distance: Then (in the last *Prob.*) the Force of a Particle at b (upon H) being as x^n , its Force in the

Direction HA will be as $\frac{a}{x} \times x^n$ or ax^{n-1} ; which

multiply'd by $2px\dot{x}$ (as before) gives $2pax^n\dot{x}$: whereof

the Fluent $\frac{2pax^{n+1} - 2pa^{n+2}}{n+1}$ ($= \frac{2p}{n+1} \times$

$AH \times BH^{n+1} - AH^{n+2}$) will be as the Force required.

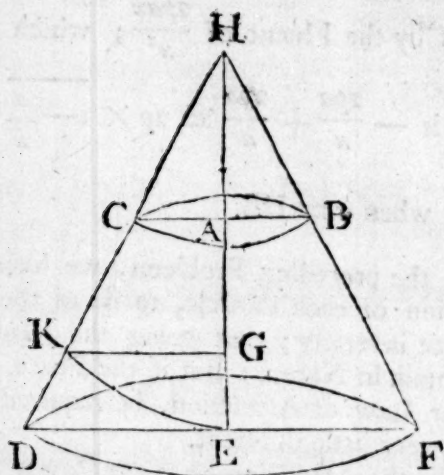
PROB. III.

377. To determine the Attraction of a Cone DHF at its Vertex; the Attraction of each Particle being as the Square of the Distance inversely.

Put the Axis $EH = a$, the Length of the Slant-Side HD (or HF) $= b$, and AH (considered as variable) $= x$: Then (by *sim. Triangles*) a (HE) : b (HF)

$:: x$

$\therefore x (HA) : HB = \frac{bx}{a}$. But, by the last Problem,



the Attraction of all the Particles in the Circle BC will be measured by $2p \times 1 - \frac{AH}{BH} = 2p \times 1 - \frac{a}{b}$ (because $HB = \frac{bx}{a}$): Which therefore being multiply'd by $\dot{x}$, and the Fluent taken, we thence have $x - \frac{ax}{b}$ for the Attraction of ACHB: And this, when $x=a$, will be, $2p \times EH - \frac{EH^2}{DH}$, the Force of the whole Cone DEHF: Which, if HK be made $= HE$, and KG perpendicular to HE, will likewise be truly defined by $2p \times EG$ (because $HG = \frac{EH^2}{DH}$). Q. E. I.

COROLLARY.

378. Seeing the Attraction of ACHB is, every where, as $x - \frac{ax}{b}$, or $\frac{b-a}{b} \times x$, it follows that the Forces of similar Cones, at their Vertices, are directly as their Altitudes.

PROB.

PROB. IV.

379. To find the Force of a Cylinder CBRF, at any Point A in the produced Axis; the Law of Attraction being still as in the preceding Problems.

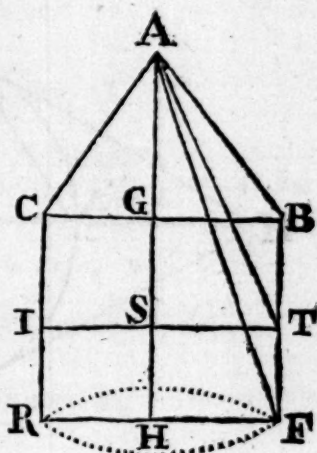
Put BG (= CG = RH) = b ; and let AS (taken as variable) = x :

Therefore $AT = \sqrt{b^2 + x^2}$,

and $2p \times 1 - \frac{AS}{AT} = 2p \times$

$1 - \frac{x}{\sqrt{b^2 + x^2}}$: Which

(by Prob. 2.) expresses the Force of all the Particles in the circular Surface IST.



Therefore $2p \times \dot{x} - \frac{x\dot{x}}{\sqrt{b^2 + x^2}}$ is the Fluxion of the

required Force: Whose Fluent ($2p \times x - \sqrt{b^2 + x^2}$)

when $x = AG$, will be $= 2p \times AG - AB$; but when

$x = AH$, it will be $= 2p \times AH - AF$: Hence, by taking the former of these Values from the latter, we

have $2p \times AB + BF - AF$ for the Measure of the true Force by which a Corpuscle at A is urged towards the Cylinder.

$\frac{p \times ax^2 - \frac{1}{3}x^3}{b^2}$ will be the Attraction of the Segment
 ABS: Which therefore, when B coincides with G and
 x is $= 2a$, becomes $\frac{4pa^3}{3b^2}$, for the Measure of the
 Attraction of the whole Sphere. Q. E. I.

COROLLARY I.

381. Hence the Attraction $\left(\frac{4pa^3}{3b^2}\right)$ at the Surface
 of the Sphere, where b is $= a$, will be $\frac{4pa}{3}$; and
 therefore is directly as the Radius of the Sphere.

COROLLARY II.

382. Since $\frac{4pa^3}{3}$ is known to express the Content of
 Sphere whose Radius is a^* , it is evident that the At- * Art. 148,
 traction $\left(\frac{4pa^3}{3b^2}\right)$ of any Sphere is, universally, as its
 Quantity of Matter directly, and the Square of the Di-
 stance from its Center inversely; and is, moreover, the
 very same as it would be, was all the Matter in the
 Sphere to be united in a Point at the Center.

COROLLARY III.

383. If instead of a Corpuscle, or a single Particle
 of Matter, at H, we suppose another Sphere, having its
 Center at H: Then, since the two Spheres, at O and
 H, act upon each other with the very same Forces, as
 if each Mass was contracted into its Center, it follows
 that the absolute Force, with which two spherical Bo-
 dies tend towards each other, is as the Product of their
 Masses directly, and the Square of the Distance of their
 Centers

Centers inverfely: And therefore, if the Maffes are given, will be barely as the Square of the Distance.

P R O B. VI.

384. To determine the same as in the laft Problem, the Force of each Particle being as any Power (n) of the Distance.

Let $HB = x$, and let every thing else remain as above; then we fhall have $y = \frac{c^2 + 2ac + x^2}{2b} = d + \frac{x^2}{2b}$ (by putting $d = \frac{c^2 + 2ac}{2b}$) and confequently $\dot{y} = \frac{x\dot{x}}{b}$.

Now the Attraction of all the Particles in the circular Surface BS, is as $\frac{2p}{n+1} \times \overline{Hn \times HB^{n+1} - Hn^{n+2}}$ (by

Art. 376.) $= \frac{2p}{n+1} \times \overline{yx^{n+1} - y^{n+2}}$: Which, multi-

ply'd by $\dot{y}$, gives $\frac{2p}{n+1} \times \overline{x^{n+1}y\dot{y} - y^{n+2}\dot{y}}$ for the Fluxion of the required Force: Which, becaufe $y\dot{y}$ is $= \overline{d + \frac{x^2}{2b}} \times \frac{x\dot{x}}{b} = \frac{dx\dot{x}}{b} + \frac{x^3\dot{x}}{2b^2}$, will likewise be expreffed

by $\frac{2p}{n+1} \times \overline{\frac{dx^{n+2}\dot{x}}{b} + \frac{x^{n+4}\dot{x}}{2b^2} - y^{n+2}\dot{y}}$: Whereof the

Fluent is $\frac{2p}{n+1} \times \overline{\frac{dx^{n+3}}{n+3 \times b} + \frac{x^{n+5}}{n+5 \times 2b^2} - \frac{y^{n+3}}{n+3}}$:

Which, when B coincides with A, or $x=y=c$, will be $=$

$\frac{2p}{n+1} \times \overline{\frac{dc^{n+3}}{n+3 \times b} + \frac{c^{n+5}}{n+5 \times 2b^2} - \frac{c^{n+3}}{n+3}}$: But, when

B co-

B coincides with G, or $x = y = 2a + c (=f)$ it will

$$\text{become} = \frac{2p}{n+1} \times \frac{df^{n+3}}{n+3 \times b} + \frac{f^{n+5}}{n+5 \times 2b^2} - \frac{f^{n+3}}{n+3} :$$

Therefore the Difference of these two, which is =

$$\frac{2pf^{n+3}}{n+1} \times \frac{n+5 \times 2bd - 2b^2 + n+3 \times f^2}{n+3 \times n+5 \times 2b^2} - \frac{2pc^{n+3}}{n+1} \times$$

$$\frac{n+5 \times 2bd - 2b^2 - n+3 \times c^2}{n+3 \times n+5 \times 2b^2} =$$

$$\frac{1+n \times ab - cc \times 2pf^{n+3} + 5+n \times ab + cc \times 2pc^{n+3}}{n+1 \times n+3 \times n+5 \times b^2}$$

(because $f = a + b$, and $2db = c^2 + 2ac$) will be the Attraction of the whole Sphere. *Q. E. I.*

COROLLARY.

385. Hence, the Attraction at the Surface of the Sphere (where $c = 0$) will be $\frac{2p}{n+1} \times$

$$\frac{1+n \times 2a^{n+3} + n+5 \times 0^{n+3}}{n+1 \cdot n+3 \cdot n+5} : \text{Which, if } n+3 \text{ be}$$

positive, will be $= \frac{2p \times 2a^{n+3}}{n+3 \times n+5}$; but, otherwise, infinite.

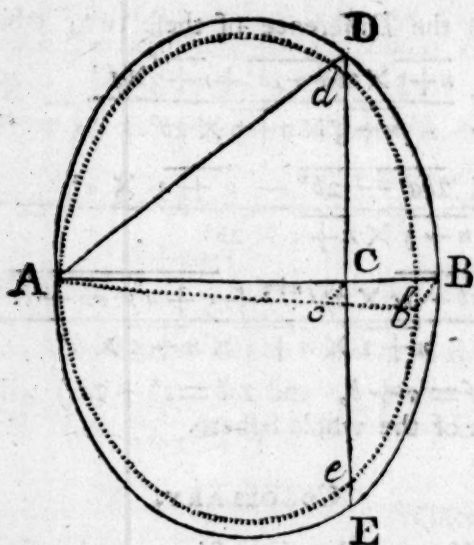
PROB. VII.

386. Supposing ADBbA to be a Cuneus of uniformly dense Matter, compriz'd by two equal and similar elliptic Planes ADBEA and AdbeA, inclin'd to each other, at the common Vertex A, of either their first or second Axes, in an indefinitely small Angle BAb; To determine the Attraction thereof at the Point A, supposing the Force of each Particle of Matter to be as the Square of the Distance inversely.

G g 3

Let

Let DE be any Ordinate to the Axis AB , and let AD be drawn; also put $AB=a$, $BC=x$, $CD=y$, and the Sine of the Angle BAb , formed by the two Planes



(to the Radius r) $= d$; and let the Equation of either Curve be $y^2 = fx - x^2 - gx^2$: Which will answer, to the Conjugate, or Transverse Axis thereof, according as the Value of g is positive or negative.

Now it will be, r (Radius) : $d :: a - x$ (AC) : Cc
 $= d \times \overline{a - x}$, the Thickness of the *Cuneus* at the Ordinate (or Section) DE : Moreover, because $AD^2 = AC^2 + CD^2$, we have $AD = \sqrt{a - x^2 + fx - x^2 - gx^2}$:

Whence, $\frac{DE \times Cc}{AC \times AD}$, expressing (by *Art.* 374.) the Attraction of the Particles in the indefinitely narrow Rectangle

$DE \times Cc$, will be defined by $\frac{2d\sqrt{fx - x^2 - gx^2}}{\sqrt{a - x^2 + fx - x^2 - gx^2}}$:

Which therefore, multiply'd by $\dot{x}$, will give the Fluxion of the Force to be found. But when $fx - x^2 - gx^2$ be-

becomes = 0, x will be = $\frac{f}{1+g}$ (=AB) = a ; therefore, by substituting for f , our Fluxion will be transformed to

$$\frac{2d\dot{x} \sqrt{1+g \times ax - 1+g \times x^2}}{\sqrt{a-x^2 + 1+g \times ax - 1+g \times x^2}} =$$

$$\frac{2d\dot{x} \sqrt{1+g \times ax - x^2}}{\sqrt{a-x^2 + 1+g \times ax - x^2}} = \frac{2d\dot{x} \sqrt{1+g \times x}}{\sqrt{a-x + 1+g \times x}} =$$

$$\frac{2d \times (1+g)^{\frac{1}{2}} \times x^{\frac{1}{2}} \dot{x}}{(a+gx)^{\frac{1}{2}}} = \frac{(1+g)^{\frac{1}{2}} \times 2dx^{\frac{1}{2}} \dot{x}}{a^{\frac{1}{2}}} \times$$

$$1 - \frac{gx}{2a} + \frac{3g^2x^2}{2 \cdot 4a^2} - \frac{3 \cdot 5g^3x^3}{2 \cdot 4 \cdot 6a^3} \text{ \&c.} \quad \text{Whereof the}$$

Fluent, when $x=a$, will be $(1+g)^{\frac{1}{2}} \times 2ad \times$

$$\frac{2}{3} - \frac{2}{5} \times \frac{g}{2} + \frac{2}{7} \times \frac{3g^2}{2 \cdot 4} - \frac{2}{9} \times \frac{3 \cdot 5g^3}{2 \cdot 4 \cdot 6} \text{ \&c.}$$

Which, because $(1+g)^{\frac{1}{2}} \times a$ is = $f \times (1+g)^{-\frac{1}{2}} = f \times$

$1 - \frac{g}{2} + \frac{3g}{2 \cdot 4} - \frac{3 \cdot 5g^2}{2 \cdot 4 \cdot 6} \text{ \&c.}$ will (by multiplying the two Serieses together \&c.) be reduced to $2df \times$

$$\frac{2}{3} - \frac{2 \cdot 4g}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6g^2}{3 \cdot 5 \cdot 7} - \frac{2 \cdot 4 \cdot 6 \cdot 8g^3}{3 \cdot 5 \cdot 7 \cdot 9} \text{ \&c.}$$

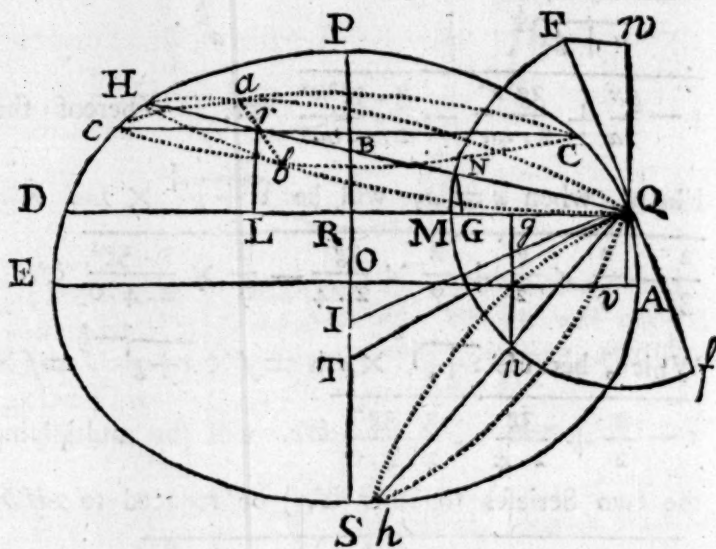
Q. E. I.

It may be observed, that the Fluent given above may be brought out without an Infinite Series (*by Art.* 126. and 278.) But the Solution here exhibited is best adapted to what follows hereafter; to which the Proposition itself is premised as a *Lemma*.

P R O B. VIII.

387. To determine the Attraction at any Point Q in the Surface of a given Spheroid $OAPES$.

Let QRL be perpendicular to the Axis PS of the Spheroid, and QT perpendicular to the Tangent Ff of the generating Ellipsis at Q , meeting PS in T : Moreover, let $QaHb$ be a Section of the Spheroid by a Plane perpendicular to that of the Ellipsis $APES$, and thro' any Point r , in the Axis thereof, draw CBc and rL parallel to AE and PS : And make the Abscissa $Qr = x$, its corresponding Semi-Ordinate ra (or rb) $= y$, $QR = a$,



and $RT = b$; also let the Sine (NG) of the Angle HQD (to the Radius $NQ = 1$) $= p$, its Co-sine $QG = q$, and the Ratio of QA^2 to OP^2 , as any given Quantity b to Unity. Now, by reason of the similar Triangles QrL and QNG , we have rL (BR) $= px$, and $QL = qx$, and therefore Br (RL) $= qx - a$: Also, from the Nature of the Ellipsis, $AO^2 : PO^2$

$$(b : 1) :: RT \ (b) : OR = \frac{b}{b} : \text{Likewise } AO^2 : PO^2$$

$$(b : 1)$$

($b:1$) :: $QR^2 : OP^2 - OR^2$; and $PO^2 : AO^2$ ($1:b$)

:: $OP^2 - OB^2 : BC^2 = b \times \overline{OP^2 - OB^2} = b \times$

$\overline{OP^2 - OR + RB^2} = b \times \overline{OP^2 - OR^2 - 2OR \times RB - RB^2}$

$= QR^2 + b \times \overline{-2OR \times RB - RB^2}$; because (by the former

Proportion) $QR^2 = b \times \overline{OP^2 - OR^2}$; Whence, by the Property of the Circle, *Cacb*, we get y^2 ($BC^2 - Br^2$) $= QR^2 -$

$Br^2 - b \times \overline{2OR \times RB + RB^2} = a^2 - \overline{qx - a^2}^2 - b \times$

$\frac{2b}{b} \times \overline{px + p^2 x^2} = \overline{aq - bp} \times 2x - \overline{q^2 + bp^2} \times x^2 :$

Which Equation, by making $1+B=b$, becomes $y^2 =$

$\overline{aq - bp} \times 2x - \overline{q^2 + p^2 + Bp^2} \times x^2 = \overline{aq - bp} \times 2x - x^2 -$

$Bp^2 x^2$ (because $q^2 + p^2 = 1 = QN^2$: Which being only of two Dimensions, the Curve $QaHb$, whereto it belongs, is an Ellipsis.

The Equation of the Curve $QaHb$ being now obtained, let its Axis QH be supposed to revolve about Q , as a Center (the Plane of the Curve being always perpendicular to that of the Ellipsis $APES$) and let the Fluxion of the Arch MN (expressing the Angle described from the time the said Axis begins its Motion at the Position

ALD) be denoted by $\dot{A}$: Then, it is evident from the preceding Problem, that, $\overline{2aq - 2bp} \times 2\dot{A} \times$

$\frac{2}{3} - \frac{2 \cdot 4 B p^2}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6 B^2 p^4}{3 \cdot 5 \cdot 7}$ &c. will be the Fluxion

of the Attraction of the corresponding Part DQH of the Solid, upon a Corpuscle at Q , considered as acting in the Direction HQ (which Expression is found, by,

barely, writing $2aq - 2bp$, $\dot{A}$, and Bp^2 , in the said Problem, for f , d , and g respectively.)

Hence,

Hence, by the Resolution of Forces, the Fluxion of the Attraction, in the Directions QR and Qw (perpendicular to QR) will be truly exhibited by $\overline{2aq - 2bp}$

$$\times 2\dot{A}q \times \frac{2}{3} - \frac{2 \cdot 4B\dot{p}^2}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6B^2\dot{p}^4}{3 \cdot 5 \cdot 7} \text{ \&c. and}$$

$$\overline{2aq - 2bp} \times 2\dot{A}p \times \frac{2}{3} - \frac{2 \cdot 4B\dot{p}^2}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6B^2\dot{p}^4}{3 \cdot 5 \cdot 7} \text{ \&c.}$$

Let now another Plane Qb be supposed to revolve about the Point Q, the contrary Way to the former, from QD towards Qf; and let (*ng*) the Sine of the Angle RQb be denoted by *P*, and its Co-sine (Qg) by *Q*: Then the Fluxion of the Attraction of the Part DQb, in the foresaid Directions QR and Qw (by writing $-\dot{P}$ instead of $\dot{p}$ and $\dot{Q}$ instead of $\dot{q}$) will appear to be

$$\overline{2aQ + 2bP} \times 2\dot{A}Q \times \frac{2}{3} - \frac{2 \cdot 4B\dot{P}^2}{3 \cdot 5} + \frac{2 \cdot 4 \cdot 6B^2\dot{P}^4}{3 \cdot 5 \cdot 7} \text{ \&c.}$$

$$\text{and } \overline{2aQ + 2bP} \times -2\dot{A}P \times \frac{2}{3} - \frac{2 \cdot 4B\dot{P}^2}{3 \cdot 5} +$$

$$\frac{2 \cdot 4 \cdot 6B^2\dot{P}^4}{3 \cdot 5 \cdot 7} \text{ \&c.}$$

Which being added to those of

the former Part, in the same Directions, and $\frac{\dot{p}}{q}$ and

$\frac{\dot{P}}{Q}$ respectively substituted instead of $\dot{A}^*$, we have

* Art 142.

$$4a \text{ into } \frac{2}{3} \times \overline{q\dot{p} + Q\dot{P}} - \frac{2 \cdot 4B}{3 \cdot 5} \times \overline{qp^2\dot{p} + Qp^2\dot{P}} \text{ \&c.}$$

$$+ 4b \text{ into } \frac{2}{3} \times \overline{P\dot{P} - p\dot{p}} - \frac{2 \cdot 4B}{3 \cdot 5} \times \overline{P^3\dot{P} - p^3\dot{p}} \text{ \&c.}$$

And

$$4a \text{ into } \frac{2}{3} \times \overline{p\dot{p} - P\dot{P}} - \frac{2 \cdot 4B}{3 \cdot 5} \times \overline{p^3\dot{p} - P^3\dot{P}} \text{ \&c.}$$

$$- 4b \text{ into } \frac{2}{3} \times \overline{\frac{p^2\dot{p}}{q} + \frac{P^2\dot{P}}{Q}} - \frac{2 \cdot 4B}{3 \cdot 5} \times \overline{\frac{p^4\dot{p}}{q} + \frac{P^4\dot{P}}{Q}} \text{ \&c.}$$

for

for the Fluxion of the Attraction of both Parts together in the forefaid Directions : Whereof the Fluents, when N coincides with F, and n with f , will be the Attraction of the whole Spheroid in thofe Directions. But now, in order to determine thefe Fluents with as little Trouble as poffible, let m be affumed to denote any

whole pofitive Number; then the Fluent of $\frac{p^{2m}\dot{p}}{q}$, or

$$\frac{p^{2m}\dot{p}}{\sqrt{1-p^2}}, \text{ will be univerfally } = \frac{-q}{2m} \times p^{\frac{2m-1}{2m-1}}$$

$$+ \frac{2m-1}{2m-2} \times p^{\frac{2m-3}{2m-2}} + \frac{2m-1 \cdot 2m-3}{2m-2 \cdot 2m-4} \times p^{\frac{2m-5}{2m-4}} (m)$$

$$+ \frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \times \text{the Arch (MN) whose Sine}$$

is p^* : And that of $\frac{p^{2m}\dot{p}}{\mathcal{Q}}$, or $\frac{p^{2m}\dot{p}}{\sqrt{1-p^2}}$ (in the * Art. 296.

$$\text{fame Manner) } = \frac{-\mathcal{Q}}{2m} \times p^{\frac{2m-1}{2m-1}} + \frac{2m-1}{2m-2} \times p^{\frac{2m-3}{2m-2}}$$

$$\&c. + \frac{1 \cdot 2 \cdot 3 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \times \text{Arch (Mn) whose Sine}$$

is P . But when N coincides with F, and n with f , the Sines p and P , of the Arches MF and Mf, becoming equal, and (the Co-fine) $\mathcal{Q} = -(\text{Co-fine}) q$,

it is evident that the Sum of the Fluents of $\frac{p^{2m}\dot{p}}{q}$ and

$$\frac{p^{2m}\dot{p}}{\mathcal{Q}}, \text{ will, in that Cafe, be truly exhibited by}$$

$$\frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \times MF + \frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \times$$

$$Mf, \text{ or its Equal } \frac{1 \cdot 3 \cdot 5 \dots 2m-1}{2 \cdot 4 \cdot 6 \dots 2m} \times FMf; \text{ be-}$$

caufe

cause, then all the rest of the Terms (by reason of the equal Quantities P , p and Q , $-q$) destroy one another. After the same Manner the Sum of the Fluents of $qp^{2m}p$ and $Qp^{2m}P$, in the foresaid Circumstance, will

* Art. 297. appear to be $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots \overline{2m-1}}{2 \cdot 4 \cdot 6 \cdot 8 \dots \overline{2m+2}} \times \text{FMf}^*.$

Now, to apply this to the Matter in hand, let the Exponent of B , in any Term of either of the above found Fluxions be, universally, expressed by n ; then the numeral Coefficient (annexed to B) will be defined

by $\frac{2 \cdot 4 \cdot 6 \dots \overline{2n+2}}{1 \cdot 3 \cdot 5 \dots \overline{2n+3}}$, and the variable Quantities

multiplied thereby, in the first Line of the former

Fluxion, will be $qp^{2n}p + Qp^{2n}P$: Therefore

$\frac{2 \cdot 4 \cdot 6 \dots \overline{2n+2}}{1 \cdot 3 \cdot 5 \dots \overline{2n+3}} \times B^n \times qp^{2n}p + Qp^{2n}P$ is a Ge-

neral Term, (from whence, if n be expounded by 1, 2, 3 &c. successively, that whole Line will be produced.) But, the Fluent of $qp^{2n}p + Qp^{2n}P$, in the Circumstance above specified, (putting $m=n$ and FMf

$= k$, appears to be $= \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots \overline{2n-1}}{2 \cdot 4 \cdot 6 \cdot 8 \dots \overline{2n+2}} \times k:$

Which, therefore, multiply'd by $\frac{2 \cdot 4 \cdot 6 \dots \overline{2n+2}}{3 \cdot 5 \dots \overline{2n+3}}$

$\times B^n$, gives $\frac{1 \cdot 3 \cdot 5 \cdot 7 \dots \overline{2n-1}}{2 \cdot 4 \cdot 6 \cdot 8 \dots \overline{2n+2}} \times \frac{2 \cdot 4 \cdot 6 \dots \overline{2n+2}}{3 \cdot 5 \dots \overline{2n+3}}$

$\times B^n k = \frac{B^n k}{2n+1 \times 2n+3}$, for the true Fluent of the

said General Term: Which, if n be expounded by 0,

0, 1, 2, 3 &c. successively, will become equal to $\frac{k}{1.3}$,

$\frac{Bk}{3.5}$, $\frac{B^2k}{5.7}$, $\frac{B^3k}{7.9}$ &c. respectively; and therefore the

Fluent of the whole Line (drawn into the general

Multiplicator 4a) is $= 4ak \times \frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} -$

$\frac{B^3}{7.9}$ &c. But now, for the Fluent of the second

Line: This, it is plain, will be $= 4b$ into $\frac{2}{3} \times$

$\frac{P^2}{2} - \frac{p^2}{2} - \frac{2.4B}{3.5} \times \frac{P^4}{4} - \frac{p^4}{4}$ &c. Which, in the

foresaid Circumstance, when $P = p$, intirely vanishes.

Therefore it appears, that the Attraction of the whole Spheroid, in the Direction QR, is truly expressed by

$4ak \times \frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} - \frac{B^3}{7.9}$, or its Equal

$4k \times \frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}$ &c. $\times QR$.

After the same Manner the Fluent of the first Line, in the latter of our two Fluxions, will be found to

vanish: And $\frac{2.4.6 \dots \overline{2n+2}}{1.3.5 \dots \overline{2n+3}} \times B^n \times \frac{p^{\overline{2n+2}} \dot{p}}{q} +$

$\frac{p^{\overline{2n+2}} \dot{p}}{2}$ will be a General Term to the second Line.

Whereof the Fluent (by expounding $2m$ by $2n+2$)

appears, from above, to be $= \frac{2.4.6 \dots \overline{2n+2}}{3.5.7 \dots \overline{2n+3}} \times$

$B^n k \times \frac{1.3.5 \dots \overline{2n+1}}{2.4.6 \dots \overline{2n+2}} = \frac{B^n k}{2n+3}$: Which, when

n is

the Composition of Forces, QI will be the Direction of the Attraction, or the Line in which a Corpuscle at Q tends to descend: And the Attraction itself, in that Direction, (being to that in QR, as QI to QR) will be

$$\text{defined by } 4k \times \frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} \mathcal{C}. \times QI;$$

which, since $4k$ is constant, will also be as $\frac{1}{1.3} -$

$$\frac{B}{3.5} + \frac{B^2}{5.7} \mathcal{C}. \times QI. \quad \text{Q. E. I.}$$

COROLLARY.

388. Since, by Construction, $RI : QR :: 1 + B \times$

$$\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9} \mathcal{C}. \times OR : \frac{1}{1.3} - \frac{B}{3.5}$$

$$+ \frac{B^2}{5.7} \mathcal{C}. \times QR, \text{ it follows that } \frac{1}{1.3} - \frac{B}{3.5} +$$

$$\frac{B^2}{5.7} \mathcal{C}. : 1 + B \times \frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \mathcal{C}. :: RO : RI;$$

$$\text{whence (by Division) } \frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} \mathcal{C}. : \frac{3B}{3.5}$$

$$- \frac{3B^2}{5.7} + \frac{3B^3}{7.9} \mathcal{C}. :: OR \left(: \frac{OT}{B} \right) : OI; \text{ and}$$

$$\text{consequently, } \frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7} \mathcal{C}. : 3 \times \frac{1}{3.5}$$

$$- \frac{B}{5.7} + \frac{B^2}{7.9} \mathcal{C}. :: OT : OI.$$

Hence it appears that the Direction QI, of the absolute Attraction, divides the Part of the Axis OT, intercepted by the Center and Normal, in a given Ratio: And that the Attraction itself (being defined

fin'd by $\frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \mathcal{E}c. \times QI$) is every
where as the said Line of Direction QI.

, SCHOLIUM.

389. Although the foregoing Conclusions are exhibited by Infinite Serieses, yet the Sums of those Serieses are explicable by means of the Arch of a Circle.

Thus, let the Series $\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \mathcal{E}c.$ (which is one of the two original ones above found) be put $= S$, and let $B = t^2$; then by Substitution, and multiplying the whole Equation by t^3 , we shall have $\frac{t^3}{3} - \frac{t^5}{5} + \frac{t^7}{7} \mathcal{E}c. = t^3 S$, and consequently $t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} \mathcal{E}c. = t - t^3 S$: Where, the former Part of the Equation is known to express the Arch of a Circle, whose

- Art. 142. Tangent is t ($B^{\frac{1}{2}}$) and Radius Unity *: Wherefore, putting that Arch $= A$, we have $A = t - t^3 S$, and consequently $S = \frac{t-A}{t^3} = \frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} \mathcal{E}c.$

Moreover, since it appears that

$$\left. \begin{aligned} &\frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7} \mathcal{E}c. \\ & - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9} \mathcal{E}c. \end{aligned} \right\} \text{is} = \frac{2B}{3 \cdot 5} - \frac{2B^2}{5 \cdot 7} + \frac{2B^3}{7 \cdot 9} \mathcal{E}c.$$

(where the Sum of $\frac{B}{3} - \frac{B^2}{5} + \frac{B^3}{7} \mathcal{E}c.$ is already found $= \frac{t-A}{t^3} \times B = \frac{t-A}{t}$, and where That
of

of $-\frac{B}{5} + \frac{B^2}{7}$ &c. by the same Method will come

out $= \frac{t - A - \frac{1}{3}t^3}{t^3}$ it is evident that $\frac{2B}{3 \cdot 5} - \frac{2B^2}{5 \cdot 7}$

$$+ \frac{2B^3}{7 \cdot 9} \text{ &c. } = \frac{t - A}{t} + \frac{t - A - \frac{1}{3}t^3}{t^3} =$$

$$\frac{\frac{2}{3}t^3 + t - A \times \overline{1 + t^2}}{t^3}; \text{ and consequently } \frac{1}{1 \cdot 3} -$$

$$\frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ &c. } \left(= \frac{1}{3} - \frac{\frac{2}{3}t^3 + t^2 - A \times \overline{1 + t^2}}{2t^3} \right)$$

$$= \frac{A \times \overline{1 + t^2} - t}{2t^3}: \text{ Which is the Value of the other}$$

original Series found above: From whence that of

$$\frac{3}{3 \cdot 5} - \frac{3B}{5 \cdot 7} + \frac{3B^2}{7 \cdot 9} \text{ will also be had } =$$

$$\frac{3t + 2t^3 - 3A \times \overline{1 + t^2}}{2t}.$$

Hence, if

$$\frac{t - A}{t^3} \left(= \frac{1}{3} - \frac{B}{5} + \frac{B^2}{7} - \frac{B^3}{9} \right) \text{ be put } = f;$$

$$\frac{A \times \overline{1 + t^2} - t}{2t^3} \left(= \frac{1}{1 \cdot 3} - \frac{B}{3 \cdot 5} + \frac{B^2}{5 \cdot 7} \text{ &c. } \right) = g;$$

And

$$\frac{3t + 2t^3 - 3A \times \overline{1 + t^2}}{2t} \left(= \frac{3}{3 \cdot 5} - \frac{3B}{5 \cdot 7} + \frac{3B^2}{7 \cdot 9} \text{ &c. } \right) = b$$

it is evident that OT will be to OI, in the constant Ratio of g to b ; and that the Forces in the Directions QI, QR, and Qv, will be as $g \times QI$, $g \times QR$, and $f \times$

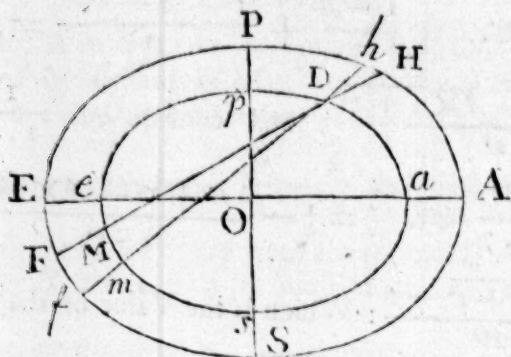
$\overline{1 + B \times OR}$ respectively: Where $1 + B$ is $= \frac{AO^2}{PO^2}$.

H h

PROB.

PROB. IX.

390. To determine the Attraction at any Point D within a given Spheroid OAPES.



Let *Oapes* be another Spheroid, concentric with, and similar to, the given one; whose Surface *DeM* &c. passes thro' the given Point *D*; also let *FDf* and *HDh* be taken as two opposite, indefinitely slender, Cones (or Pyramids) conceived to be form'd by drawing innumerable Lines *HDF*, *hDf* &c. through the common Vertex *D* which Cones (or Pyramids) having the same Angle, may be considered as similar; and so their Forces, at *D*, will be as the Altitudes *DF* and *DH**: And, therefore, the Excess of the former, above the latter, or the Force whereby a Corpuscle at *D*, tends towards *F*, through the, contrary, Action of the two opposite Cones, will be as *DF—DH*, or as *DM*; because (by the Property of the Ellipsis) *MF* is, in all Positions, equal to *DH*.

* Art. 378. Hence it appears that the Parts of Matter *FMmf* and *HDh*, without the Spheroid *apes* (acting equally, in contrary Directions) can have no Effect at *D*: And this, being every where the Case, the whole, efficacious, Force at *D* must therefore be that of the Spheroid *Oapes*.

Hence, if the Ratio of Oa^2 to Op^2 (or of OA^2 to OP^2) be denoted by that of $1+B$ to 1 , as in the last Problem,

it follows, from thence, that the Attraction at D, in the Directions DM and DN (perpendicular to PS and AE;

see the next Fig.) will be expounded by $\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}$

&c. $\times$ DM, and $1 + B \times \frac{1}{3} - \frac{B}{.5} + \frac{B^2}{.7} - \frac{B^3}{9}$

&c. $\times$ DN respectively, or by their Equals $g \times$ DM

and $f \times 1 + B \times$ DN : Where the Values of f and g are the same as given in the preceding Article.

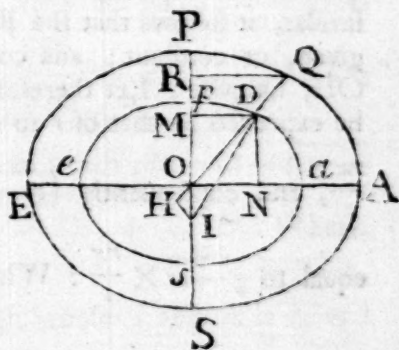
COROLLARY.

391. Hence the Force wherewith a Corpuscle, any where within a given Spheroid, is attracted, either, towards the Axis, or the Plane of its Equator, is directly as the Distance therefrom.

PROB. X.

392. Supposing every Particle of Matter in a Spheroid to have a Tendency to recede, both, from the Axis PS, and from the Plane of the greatest Circle, by Means of Forces that are as the Distances from the said Axis, and Plane, respectively; to find the Direction DI wherein a Corpuscle, at any Point D, tends to move through the Action of the said Forces and the Attraction conjunctly; and likewise the whole compound Force in that Direction.

Let DM and DN be perpendicular to PS and AE, and let the given Forces, in the Direction of those Lines (independent of the Attraction) be expressed by $m \times$ DM and $n \times$ DN respectively.



Therefore, since (by the last Problem) the Force of Attraction in the said Directions is defined by $g \times DM$ and $f \times \overline{1+B} \times DN$, the whole resulting Forces will be truly denoted by $\overline{g-m} \times DM$, and $f \times \overline{1+B-n} \times DN$: Whence (by the Composition of Forces) it will be, $\overline{g-m} : f \times \overline{1+B-n} :: DN \text{ (OM)} : MI$; whence the Point I is given:

Also $DM : DI :: \overline{g-m} \times DM \text{ (the Force in the Direction DM)} : \overline{g-m} \times DI$, the Force in DI. *Q.E.I.*

P R O B. XI.

393. *Every thing being supposed as in the preceding Problems, 'tis required to determine the Force of all the Particles in the Line (or Column) QDO tending to the Center O of the Spheroid.*

Let IH be perpendicular to QO produced (*see the last Fig.*) then the absolute Force, in the Direction DI, being $\overline{g-m} \times DI$, that in the Direction DH, whereby a Corpuscle at D is urged towards the Center, will be $\overline{g-m} \times DH$. Let now OD (considered as variable) be denoted by x ; then because the Ratio of OM to MI

is given (being every where as $\overline{g-m}$ to $f \times \overline{1+B-n}$, by the Precedent) and the Triangles ODM and IOH are similar, it follows that the Ratio of OD to OH will be given, or constant; and consequently that of DH to OH, likewise: Let therefore this Ratio of DH to OH be expressed by that of r to s , and we shall have $DH =$

$\frac{rx}{s}$, and consequently $(\overline{g-m} \times DH)$ the Force at D,

equal to $\overline{g-m} \times \frac{rx}{s}$: Which therefore being multi-

ply'd

ply'd by $\dot{x}$, and the Fluent taken, there comes out

$$\frac{g-m}{2s} \times rx^2 = \frac{g-m}{2} \times DO \times DH, \text{ for the whole}$$

Force of the Line or Column OD at the Center.

Q. E. I.

COROLLARY.

394. If the given Forces m and n be such that the Ratio of OM to MI, (which is found to be universally as $g-m$ to $f \times \overline{1+B-n}$) may become as $1:1+B$ (or as $pO^2 : aO^2$) it is evident (from the Property of the Ellipsis) that the Line of Direction DI will be always perpendicular to the Surface of the Spheroid *Oapes*. In which Case $OD \times DH$ is also (by the Nature of the Ellipsis) $= Oa^2$: And therefore the Force $\left(\frac{g-m}{2} \right.$

$\times OD \times DH$) of OD is $= \frac{g-m}{2} \times Oa^2$: Which,

when D coincides with Q, will become $\frac{g-m}{2} \times AO^2$; and is, therefore, a constant Quantity.

Moreover, since in this Case, $g-m : f \times \overline{1+B-n}$

$:: 1 : 1+B$ (by Hypothesis) we have $m - \frac{n}{1+B} = g - f$: Which Equation, if n be taken $= 0$, gives

$$m = g - f = \frac{2B}{3 \cdot 5} - \frac{4B^2}{3 \cdot 7} + \frac{6B^3}{7 \cdot 9} \&c. = \frac{\overline{3+t^2} \times A - 3t^*}{2t^3}; \text{ * Art. 389.}$$

But, if m be taken $= 0$, it will then give $n = -\overline{1+B}$

$$\times g - f = -\overline{1+B} \times \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9} \&c. \text{ Where,}$$

$t = B^{\frac{1}{2}}$, and A = the Arch whose Tangent is t , and Radius Unity.

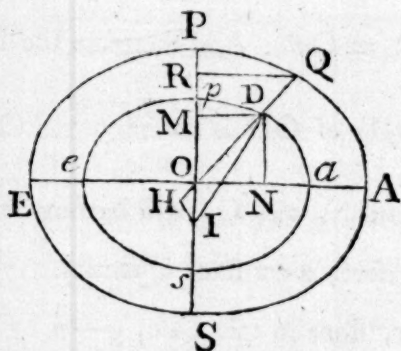
PROB. XII.

395. If an oblate Spheroid OAPES, whereof the Square of the Equatoreal Diameter AE, is to that of the Axis PS, in any given Ratio of $1+B$ to 1, revolves about its Axis, in such a Time, that the centrifugal Force, at the Equator A, is to the Attraction at the Surface of

a Sphere whose Radius is OA, in the Ratio of $\frac{2B}{3.5} -$

$\frac{4B^2}{5.7} + \frac{6B^3}{7.9}$ &c. to $\frac{1}{3}$: I say, in that Case, every

Particle of the Spheroid will be in Equilibrio; so that, tho' the Cohesion of the Parts was to cease, the Figure itself would remain unchanged.



For, the Attraction of the Spheroid, at A, being de-

fined by $\frac{1}{1.3} - \frac{B}{3.5} + \frac{B^2}{5.7}$ &c. $\times AO$ (Art. 387.)

it is evident (by conceiving $B=0$) that $\frac{AO}{3}$ will re-

present the Attraction at the Surface of the Sphere whose Radius is AO: Whence (by Hypothesis) the

centrifugal Force at A (putting $m = \frac{2B}{3.5} - \frac{4B^2}{5.7} + \frac{6B^3}{7.9}$ &c.) will be truly defined by $m \times AO$; and con-

sequently

sequently That, at any other Point D, by $m \times DM$ (because the centrifugal Forces of Bodies describing unequal Circles, in equal Times, are known to be directly as the Radii *.) Hence, and from the Corollary to the last * Art. 213. Problem, it appears that the Direction of Gravitation DI is always perpendicular to the Surface *apes*; and that the Force of all the Particles in the Line (or Canal) OD or OQ, towards the Center O, will continue invariable, take the Point Q in what Part of the Arch APE you will: From which last Consideration, it follows that the Force, or Pressure of every Canal QO, at the Center O, (considering the Body in a fluid State) will be the same: Whence (by the Principles of Hydrostatics) a Corpuscle at D has no Tendency to move, either Way, in the Line OQ: And therefore, as it hath no Tendency to move in the Direction of the Surface Dpe (the Gravitation being perpendicular thereto) it is evident, from *Mechanics*, that no Motion at all can ensue, in any Direction. Q. E. D.

COROLLARY I.

396. Since m is $= \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9}$ &c. the Gravitation ($g - m \times DI$) at any Point D in the Spheroid will therefore be as $\frac{1}{3} - \frac{B}{5} + \frac{B^2}{7}$ &c.
 $\times DI = \frac{t - A}{t^3} \times DI$ (see Art. 389.)

COROLLARY II.

397. If the Time of Revolution be given $= p$, and q be put to denote the Time wherein a (solid) Sphere, of the same Density with the Spheroid, must revolve; so that the centrifugal Force, at the Equator thereof, may be equal to the Gravity: Then, as this last Time is known to continue the same, whatever the Magnitude of that Sphere is ‡; and the centrifugal Forces, in equal ‡ Art. 213. and 381.
 H h 4 Circles,

Circles, are also known to be inverfely as the Squares of the periodic Times—it follows, that $p^2 : q^2 :: \frac{1}{3}AO$ (the Attraction, or centrifugal Force, refpe&shying the

Sphere OA, revolving in the Time q) : $\frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7}$

+ $\frac{6B^3}{7 \cdot 9} \mathcal{E}c.$ $\times AO$, the centrifugal Force of the

Spheroid at A, revolving in the Time p . From which

Proportion we get $\frac{q^2}{3p^2} = \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9} \mathcal{E}c. =$

$\frac{3 + t^2 \times A - 3t}{2t^3}$ (Art. 394.) Whence, by Help of the

Trigonometrical-Canon, the Value of t ($= B^{\frac{1}{2}}$) and, consequently, the Ratio of the two principal Diameters, will be found ; fo that all the Parts of the Spheroid

may remain in *Equilibrio*. But, when $\frac{q^2}{3p^2}$ is fmall,

the Solution by an Infinite Series is preferable : For, then

the Series $\frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} \mathcal{E}c. (= \frac{q^2}{3p^2})$ converging fufficiently fwift, we fhall, by the Reverfion thereof, find

$B = \frac{5q^2}{2p^2} + \frac{25 \times 6q^4}{4 \times 7p^4} + \frac{125 \times 37q^6}{8 \times 49p^6} \mathcal{E}c.$ In which

Cafe the Ratio of the Equatoreal Diameter to the Axis, if we take only the firft Term of the Series, will be, as

$\sqrt{1 + \frac{5q^2}{2p^2}} : 1$, or as $1 + \frac{5q^2}{4p^2}$, nearly.

Which, if $\frac{p^2}{q^2} = 289$, or the centrifugal Force at

the Equator be to the Gravity as 1 to 289 (that being
* Art. 217. the Proportion at the Equator of the Earth *) will come out as 231 to 230,

Co-

COROLLARY III.

398. Because, $\frac{3+t^2 \times A - 3t}{2t^3}$, the latter Part of

our foregoing Equation will be equal to Nothing, both when t is Nothing and Infinite, it is evident that the Value thereof cannot, in any intermediate Circumstance of t , exceed a certain assignable Quantity.

Wherefore, to determine this Limit of the Value of $\frac{q^2}{3p^2}$ (beyond which the Problem becomes impossible)

let the Fluxion of $\frac{3+t^2 \times A - 3t}{2t^3}$, or its Double

$\frac{3+t^2 \times A}{t^3} - \frac{3}{t^2}$ be taken and put $= 0$, and you will

have $-\frac{9}{9+t^2} \times \dot{A}t + \frac{3t+t^3}{t^3} \times \dot{A} + 6ti = 0$:

Which, because $\dot{A} = \frac{\dot{t}}{1+t^2}$ * will be reduced to $9t^2$ Art. 142.

$+ 7t^3 - \frac{1}{1+t^2} \times 9+t^2 \times A = 0$; where t is found $= 2,5293$, from whence the corresponding Values of

$\sqrt{1+t^2}$, and $\frac{q}{p}$ come out $= 2,7198$, and $0,5805$

&c. respectively. Hence it appears that it is impossible for the Parts of the Spheroid, in a fluid State, to continue at Rest among themselves, when the Time of

Revolution is so great that $\frac{q}{p}$ exceeds $0,5805$ &c.

And that, of all the Spheroids which can be assumed by a Fluid revolving about an Axis, That whose Equatoreal Diameter is to its Axis as $2,7198$ to Unity, will perform its Revolutions in the shortest Time.

Thus, for Example, if a (solid) Sphere of the same common Density with the Earth was to revolve about its Axis in the Time of $84\frac{3}{4}$ Minutes, the centrifugal Force

Force at the Equator thereof would, it is known, be

* Art. 217. equal to the Gravity*: Therefore, by taking $\frac{84\frac{1}{2}}{p} \left(= \frac{q}{p} \right)$

$= 0,5805$ &c. the Time p will come out =

$\frac{M}{146}$ or $2 \frac{H}{26}$. Which Time is the least, possible, wherein a Fluid, of the same common Density with the Earth, can revolve, so as to preserve its spheroidal Figure. And this holds universally, let the Magnitude of the Body, or Fluid, be what it will.

COROLLARY IV.

399. Hence also may be determined the Spheroid, which a spherical Body (of Ice or any other Matter) revolving in a given Time s , will converge to, when reduced to a fluid State.

For, since the Momenta of Rotation, in equal Spheres and Spheroids, are to one another, in a Ratio compounded of the direct Ratio of their Equatoreal Diameters, and the inverse Ratio of the Times of their Rotation, it follows, if d be put = the Diameter of the given Sphere, and E = the Equatoreal Diameter of

the required Spheroid, that $\frac{d}{s} = \frac{E}{p}$ (because the Quan-

tity of Motion about the Axis is not affected by the Action of the Particles one upon another, while the Figure of the Fluid is changing.) Moreover, since the Masses of the Sphere and Spheroid are also equal to each other (by Hypothesis) we have $d^3 (=AE^2 \times PS) =$

$\frac{E^3}{1+t^2}^{\frac{1}{2}}$: From which two Equations, exterminating

d , there arises $p = \sqrt{1+t^2}^{\frac{1}{6}} \times s$, for the Time of Revolution of the required Spheroid: Whence, by sub-

stituting this Value of p in the general Equation $\frac{q^2}{3p^2} =$

$$= \frac{3+t^2 \times A - 3t}{2t^3}, \text{ we get } \frac{q^2}{3s^2} = \frac{1+t^2}{1+t^2}^{\frac{1}{3}} \times$$

$$\frac{3+t^2 \times A - 3t}{2t^3}; \text{ from the Solution of which the}$$

Value of t , and the Spheroid itself, will be given.

But, since the Value of the latter Part of the Equation can never exceed a certain assignable Quantity, the Matter proposed can therefore be only possible under certain Limitations: In order to determine these Limi-

$$\text{tations, let the Fluxion of } \frac{1+t^2}{1+t^2}^{\frac{1}{3}} \times \frac{3+t^2 \times A - 3t}{2t^3}$$

be taken and put $= 0$, and it will be found that

$$t^4 + 24t^2 + 27 \times A - 15t^3 - 27t = 0: \text{ Whence } t$$

comes out $= 7.5$, and the corresponding Value of

$$\frac{q}{s} = 0.927, \text{ nearly.}$$

Hence the Parts of the Fluid cannot possibly come to an Equilibrium among themselves, when the Time

s is less than $\frac{q}{0.927}$, but will continue to recede from the Axis, *in Infinitum*.

If q be taken $= 84 \frac{3}{4}$ (as in the Example to the preceding Corollary) s will be equal $91 = 1:31$. From which it appears, that, if the Earth (or a spherical Body of the same Density) was to revolve

about its Axis in less than $1:31$; and, in the mean time, be reduced to a State of Fluidity, the Parts thereof towards the Equator would ascend, and continue to recede from the Axis, *in Infinitum*.

COROLLARY V.

400. Seeing the Values of t and A are given when the Spheroid is given, it follows that the Gravitation

tation $\left(\frac{t-A}{t^3} \times QI\right)$ at any Point in the Surface of a Spheroid, whereof the Parts are kept in *Equilibrio*, by their Rotation about the Axis, will be accurately as a Perpendicular to the Surface at that Point, continued to the Axis of the Figure. Therefore the Gravitation at the Equator is to that at either of the Poles, as the Equatoreal Diameter to the Axis inversly.

COROLLARY VI.

401. But, if the Spheroid differs but little from a Sphere, the Excess of *QI* above *AO* will (by the Property of the Ellipsis) be nearly as OR^2 . Whence it appears that the Increase of Gravitation, in going from the Equator to the Pole, is as the Square of the Sine of Latitude, nearly.

COROLLARY VII.

402. Moreover, since the Ratio of the Equatoreal Diameter to the Axis is found, in this Case, to be that

† Art. 397. of $1 + \frac{5q^2}{4p^2}$ to 1 †, the Excess of that Diameter above the

Axis will be to the Axis as $\frac{5q^2}{4p^2}$ to Unity; that is, as $\frac{5}{4}$

of the centrifugal Force at the Equator to the mean Force of Gravity. Whence, as the centrifugal Forces, in unequal Circles, are universally as the Radii directly, and the Squares of the periodic Times inversly, it follows that the foresaid Excess (in Figures nearly spherical) will be as the Radii directly, and as the Density and the Square of the Time of Rotation inversly: From which Proportions, the Ratios of the greatest and least Diameters of the Planets may be inferred from each other; supposing the Times of their Rotation, about their Axes, to be known.

P R O-

P R O B. XIII.

403. To determine the Figure which a Fluid will acquire when, besides the mutual Gravitation of the Parts thereof, it is attracted by another Body, so remote, that all Lines drawn from it to the Surface of the Fluid, may be taken as Parallels.

Let OAPES be the proposed Fluid, and let MPS and MQq be Right-lines, drawn from the remote Body M ; whereof the former MPS passes thro' the Center of Gravity O : Moreover, let the Plane AE be perpendicular to the Axis MOS; and put $NQ = a$ and OM (the Distance of the remote Body) $= d$; also put the Semi-diameter of the Body (at M) $= r$, and let its Density be to that of the Fluid APES, as any Quantity v to Unity.



Then since, according to the foregoing Calculations, the Attraction at the Surface of a Sphere (of a given Density) is expressed by $\frac{1}{3}$ of the Radius, it follows that the Attraction of the Body M , at its Surface,

will be explicable by $\frac{vr}{3}$: And therefore, the Force varying according to the Square of the Distance in-

versly *, it will be, $d^2 (MN^2) : r^2 :: \frac{vr}{3} : \frac{vr^3}{3d^2}$, the * Art. 382.

Attraction of M , at the Distance MN : Also $\frac{vr}{d-a}$

$(MQ^2) : r^2 :: \frac{vr}{3} : \frac{vr^3}{3 \times d-a}$, its Attraction at the

Distance

Distance MQ. Whence the Difference of these two,

$$\text{or } \frac{vr^3}{3 \times d - a^2} - \frac{vr^3}{3d^2} (= \frac{vr^3}{3d^3} \times 2a + \frac{3a^2}{d} + \frac{4a^3}{d^2}$$

&c.) will be as the Force whereby a Corpufcle at Q endeavours to recede from the Plane AE: Which be-
cause (by Hypothefis) d is very great in refpect of a ,
will (by rejecting all the Terms after the firft) be ex-

preffed by $\frac{2vr^3}{3d^3} \times a$, or its Equal $\frac{2vr^3}{3d^3} \times NQ$.

In the very fame Manner, the Force whereby a Corpufcle at q , below the Plane AE, tends to recede

therefrom, will be defined by $\frac{2vr^3}{3d^3} \times Nq$.

Now, therefore, feeing thefe Forces are, every where, as the Distances NQ , Nq , from the Plane AE, it appears (by Art. 393. and 394.) that the Figure OAPES will be a Spheroid; whereof the Equation, for the Relation of

its two principal Diameters (putting $n = \frac{2vr^3}{3d^3}$) is $n =$

$$- \frac{1}{1+B} \times \frac{2B}{3 \cdot 5} - \frac{4B^2}{5 \cdot 7} + \frac{6B^3}{7 \cdot 9} \text{ \&c. (In which,$$

the Ratio of PS^2 to AE^2 is denoted by that of 1 to $1+B$.) Hence, by reverting the Series, we have $B =$

$$- \frac{15n}{2} - \frac{225n^2}{28} \text{ \&c. and consequently } PS : AE :: 1 :$$

$$\sqrt{1 - \frac{15n}{2} - \frac{225n^2}{28}} \text{ \&c. } :: 1 : 1 - \frac{15n}{4}, \text{ nearly:}$$

Which, by reftoring the Value of n , becomes $PS : AE$

$$:: 1 : 1 - \frac{5vr^3}{2d^3}.$$

Q. E. I.

Co-

COROLLARY.

404. Because $\frac{r}{d}$ expresses the Sine of the apparent Semi-diameter of the Body M , to the Radius 1) seen at the Distance OM , it follows, if the said Sine be denoted by c , that $PS : AE :: 1 : 1 - \frac{5v}{2} \times c^3$; and consequently, by Division, $PS : PS - AE :: 1 : \frac{5v}{2} \times c^3$.

Hence it appears, that the Forces of the Planets, to produce Tides at the Earth's Surface, are to one another as their Densities, and the Cubes of their apparent Diameters conjunctly. (For the Sines of small Arcs are nearly as the Arcs themselves.)

E X A M P L E.

405. If c be taken = the Sine of $16'$ (expressing the mean Apparent Semi-diameter of the Moon) and $v = \frac{5}{4}$ (the Ratio of her Density with respect to that of the Earth) our last Proportion will become $PS : PS - AE :: 1 : 0,000000315$: Whence, if PS be taken = 42000000 Feet (the Measure of the Earth's Diameter) $PS - AE$ will come out = $13,23$.

SECT.

SECTION X.

Of the Application of FLUXIONS to the Resolution of such Kinds of Problems DE MAXIMIS ET MINIMIS, as depend upon a particular Curve, whose Nature is to be determined.

I SHALL begin this Section with premising the following useful

THEOREM.

406. *If the Relation of two flowing Quantities y and u be required; so that, when the Fluent of $y^m u$ becomes equal to a given Value, that of $\frac{y^r \times \overline{uu \pm yy}^n}{y^{2n-1}}$ may be a Maximum or a Minimum; I say, their Relation must be such that $\frac{y^{r-m} u \times \overline{uu \pm yy}^{n-1}}{y^{2n-1}}$ may be, every where, the same, or equal to a constant Quantity.*

The Demonstration hereof depends upon the subsequent

LEMMA.

407. If $ax + b\beta = 2$, wherein a and β are indeterminate, the Value of $A \times \overline{ax \pm pp}^n + B \times \overline{\beta\beta \pm pp}^n$ will be a Maximum or Minimum, when $\frac{Aa}{a} \times \overline{ax \pm pp}^{n-1}$ and $\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}^{n-1}$ are equal to each other

other. For, by taking the Fluxions of both Expressions we have $a\dot{\alpha} + b\dot{\beta} = 0$, and $2nAa\dot{\alpha} \times \overline{aa \pm pp}^{n-1} + 2nB\beta\dot{\beta} \times \overline{\beta\beta \pm pp}^{n-1} = 0$: From whence, $\dot{\alpha}$ and $\dot{\beta}$ being exterminated, there results $\frac{Aa}{a} \times \overline{aa \pm pp}^{n-1} = \frac{B\beta}{b} \times \overline{\beta\beta \pm pp}^{n-1}$. Q. E. D.

Hence, if $aa + b\beta + c\gamma + d\delta \&c. = Q$ (where $a, \beta, \gamma, \delta \&c.$ are indeterminate) it follows that $A \times \overline{aa \pm pp}^n + B \times \overline{\beta\beta \pm pp}^n + C \times \overline{\gamma\gamma \pm pp}^n + D \times \overline{\delta\delta \pm pp}^n \&c.$ will be a *Maximum* or *Minimum*, when all the Quantities $\frac{Aa}{a} \times \overline{aa \pm pp}^{n-1}$,

$\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}^{n-1}$, $\frac{C\gamma}{c} \times \overline{\gamma\gamma \pm pp}^{n-1} \&c.$ are equal

to each other. For that Expression is a *Maximum* (or *Minimum*) when it cannot be increased (or decreased) by altering the Values of the indeterminate Quantities involved therein; but it may be increased (or decreased) by altering only two of them (as a and β) whilst the rest remain unchanged; unless $\frac{Aa}{a} \times \overline{aa \pm pp}^{n-1}$ and

$\frac{B\beta}{b} \times \overline{\beta\beta \pm pp}^{n-1}$ are equal to each other. (This is

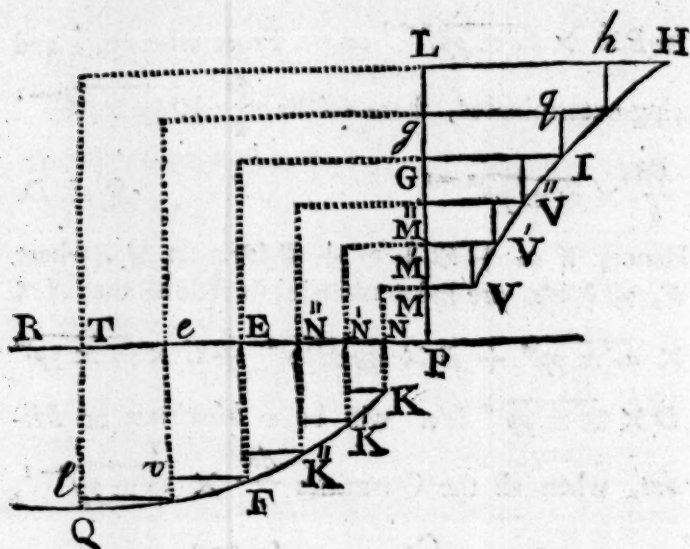
proved above.) Therefore, when $A \times \overline{aa \pm pp}^n + B \times \overline{\beta\beta \pm pp}^n + C \times \overline{\gamma\gamma \pm pp}^n + \&c.$ is a *Maximum* or *Minimum*, the Quantities $\frac{Aa}{a} \times \overline{aa \pm pp}^{n-1}$ and $\frac{B\beta}{b}$

$\times \overline{\beta\beta \pm pp}^{n-1}$ cannot be unequal: And, by the very same Argument, no other two of the Quantities above specified can be unequal.

I i

If,

If, in the Right-line PR, there be now assumed $NN' = \alpha$, $NN'' = \beta$, &c. and upon these, as Bases,



Rectangles $\dot{N}K$, $\ddot{N}K$ be supposed, whose Altitudes NK , $\dot{N}K$ &c. are denoted by a, b, c, d &c. it is evident that $a\alpha + b\beta + c\gamma + d\delta$ &c. ($= 2$) will be expressed by the Sum of all the said Rectangles, or the whole Polygon NZ .

Moreover, if, in the Right-line PL (perpendicular to PR) there be taken MM' , MM'' &c. each equal to p , and, upon these equal Bases, Rectangles MV , $M'V$ &c. be constituted, whose Altitudes are denoted by

$$A \times \frac{\overline{ax \pm pp}^n}{p^{2n}}, B \times \frac{\overline{\beta\beta \pm pp}^n}{p^{2n}}, \text{ \&c. it is likewise}$$

plain that the Value of $\frac{A \times \overline{aa \pm pp}^n}{p^{2n-1}} + \frac{B \times \overline{bb \pm pp}^n}{p^{2n-1}}$

$$+ \frac{C \times \overline{\gamma\gamma \pm p\bar{p}}^n}{p^{2n-1}}$$
 will be truly represented by the

whole

whole Polygon Mb. Which Polygon (as p is constant) will be a *Maximum* or *Minimum*, when $A \times$

$\overline{aa \pm pp}^n + B \times \overline{\beta\beta \pm pp}^n + \mathcal{E}c.$ is a *Maximum* or *Minimum*; that is, when all the Quantities $\frac{Aa}{a} \times$

$\frac{\overline{aa \pm pp}^{n-1}}{p^{2n-1}}, \frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}^{n-1}}{p^{2n-1}}, \mathcal{E}c.$ are equal to each other (as has been proved above.)

Let now, $A, B, C, D \mathcal{E}c.$ be expounded by any Powers, ($MP^r, MP^r, MP^r \mathcal{E}c.$) of the respective Distances from a given Point P ; and let, at the same time, the corresponding Values of $a, b, c, d \mathcal{E}c.$ be interpreted by any other proposed Powers $MP^m, MP^m, MP^m \mathcal{E}c.$ of the same given Distances: Then the

Area of the Polygon Nl will be expressed by $MP^m \times a + MP^m \times \beta + MP^m \times \gamma \mathcal{E}c. (= \mathcal{Q})$; and that of the

Polygon Mb , by $MP^r \times \frac{\overline{aa \pm pp}^n}{p^{2n-1}} + MP^r \times \frac{\overline{\beta\beta \pm pp}^n}{p^{2n-1}} + MP^r \times \frac{\overline{\gamma\gamma \pm pp}^n}{p^{2n-1}} + \mathcal{E}c.$ And the forefaid equal

Quantities $\frac{Aa}{a} \times \frac{\overline{aa \pm pp}^{n-1}}{p^{2n-1}}, \frac{B\beta}{b} \times \frac{\overline{\beta\beta \pm pp}^{n-1}}{p^{2n-1}} \mathcal{E}c.$

will become $MP^{r-m} \times \frac{a \times \overline{aa \pm pp}^{n-1}}{p^{2n-1}}, MP^{r-m}$

$\times \frac{\beta \times \overline{\beta\beta \pm pp}^{n-1}}{p^{2n-1}}, \mathcal{E}c.$ respectively.

Now let the Number of the Rectangles be supposed indefinitely great, and their Breadths indefinitely small,

so that the Area of each of the two Polygons Nl and Mb may be taken for that of its circumscribing Curve: Moreover, let u and y be put to represent the Distances of any two corresponding Ordinates EF and GI from the given Point P ; and let $\dot{x}$ be every where expressed

by p ($=MM=\dot{MM}=\mathcal{E}c.$) Then, u being a general

Value for any of the Quantities $\alpha, \beta, \gamma, \delta \mathcal{E}c.$ (or NN ,

$NN \mathcal{E}c.$) it follows; First, that the Fluxion of the Area of the Curve $NEFK$ (the Ordinate being, every

where, $=y^m$) will be truly defined by $y^m \dot{u}$; Secondly, that the Fluxion of the Area $MGIV$ (by substituting y, u and $\dot{y}$ instead of their Equals) will be

$\frac{y^r \times \overline{uu \pm yy}^n}{y^{2n-1}}$; and, lastly, that the Value of each

of the equal Quantities, $MP^{r-m} \times \frac{\alpha \times \overline{\alpha\alpha \pm pp}^{n-1}}{p^{2n-1}}$,

$MP^{r-1} \times \frac{\beta \times \overline{\beta\beta \pm pp}^{n-1}}{p^{2n-1}}$, $\mathcal{E}c.$ above specified, will

be expressed by $\frac{y^{r-m} \times u \times \overline{uu \pm yy}^{n-1}}{y^{2n-1}}$. Whence

the Theorem is manifest.

408. If R and S be assumed to denote any Functions of y (that is, any two Quantities expressed in Terms of y and given Coefficients;) then, in order to have the

Fluent of $S \times \frac{\overline{uu \pm yy}^n}{y^{2n-1}}$ a Maximum or Minimum,

when that of Ru becomes equal to a given Value, it is

requisite that $\frac{Su}{R} \times \frac{\overline{uu \pm yy}^{n-1}}{y^{2n-1}}$ should be a constant

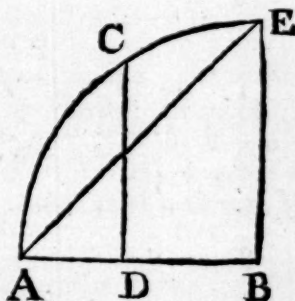
Quan-

Quantity: This, also, is evident from the preceding Demonstration; and may be of Use when the above premised Theorem is not sufficiently general.

PROB. I.

409. To determine the Nature of the Curve ACE; so that, the Length of the Arch AE being given, the Area ABE shall be a Maximum.

Calling (as usual) the Abscissa AD, x ; the Ordinate DC, y ; and the Arch AC, z , we have $\dot{x} = \sqrt{\dot{z}^2 - \dot{y}^2}$; and therefore $y\dot{x} \dagger = y \times \overline{zz - yy}^{\frac{1}{2}}$ = the Fluxion of the Area ADC. Now, since, by the Question, the



• Art. 135.

† Art. 112.

Fluent of $y \times \overline{zz - yy}^{\frac{1}{2}}$ is to be a Maximum, when That of z becomes equal to a given Quantity (ACE) let these two Fluxions be, respectively, compared with

$$\frac{y^r \times \overline{uu - yy}^n}{y^{2n-1}} \text{ and } y^m u \text{ (as given in the foregoing$$

Theorem *.) By which means, $n = \frac{1}{2}$, $r = 1$, $u = \dot{z}$, • Art 406;

and $m = 0$; and consequently $\frac{y^{r-m} u \times \overline{uu - yy}^{n-1}}{y^{2n-1}}$

$= y\dot{z} \times \overline{zz - yy}^{-\frac{1}{2}}$: Which (according to the said Theorem) being, every where, equal to a constant Quantity, we shall, by putting that Quantity = a ,

and ordering the Equation, get $\dot{z}^2 = \frac{a^2 \dot{y}^2}{a^2 - y^2}$, and $\dot{z}$

$$(\sqrt{\dot{z}^2 - \dot{y}^2}) = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}; \text{ and, consequently, (by}$$

I i 3

taking

486 Of Problems De Maximis & Minimis

taking the Fluent) $x = a - \sqrt{a^2 - y^2}$, or $2ax - xx = y^2$; which is the common Equation of a Circle.
Q. E. I.

COROLLARY.

410. It follows from hence, that the greatest Area that can possibly be contain'd by a Right-line (AE) joining two given Points, and any Curve-line ACE of a given Length, terminating in the same Points, will be when the said Curve-line is an Arch of a Circle.

PROB. II.

411. The Length of the Arch AE (see the preceding Figure) being given, to determine the Nature of the Curve, so that the Solid generated by the Rotation thereof may be a Maximum.

• Art. 145. Since the Fluent of $y^2 \times \sqrt{z^2 - y^2}^{\frac{1}{2}}$ ($= y^2 z^*$) is required to be a Maximum, when that of z has a given Value ACE, every thing will remain as in the last Problem; only, r must here be $= z$: And therefore (by the Theorem) we have $y^2 z \times \sqrt{z^2 - y^2}^{-\frac{1}{2}} = a$.
Whence $z = \frac{ay}{\sqrt{a^2 - y^4}}$; and consequently z^* ($= \sqrt{z^2 - y^2}$) $= \frac{y^2 y}{\sqrt{a^2 - y^4}}$: Which Values, if b^2 be put $= a$ (in order to have the Powers homologous) will become $z = \frac{b^2 y}{\sqrt{b^4 - y^4}}$ and $z^* = \frac{y^2 y}{\sqrt{b^4 - y^4}}$:
Whence z and z^* will be known.
Q. E. I.

PROB. III.

412. The Superficies generated by the Arch of a Curve, in its Rotation, about its Axis, being given; to determine the Curve, so that the Solid, itself, may be a Maximum.

• Art. 145. Because the Fluent of $y^2 \times \sqrt{z^2 - y^2}^{\frac{1}{2}}$ * is to be a Maximum, when that of yz becomes equal to a given Quan-

Quantity; let the Fluxions here exhibited be therefore

compared with $\frac{y^r \times \overline{uu \pm yy}^n}{y^{2n-1}}$ and $y^m \dot{u}$ (given in the

Theorem.) By means whereof (r being $= 2$, $\dot{u} = \dot{z}$,

$n = \frac{1}{2}$, and $m = 1$) we have $y\dot{z} \times \overline{z^2 - y^2}^{-\frac{1}{2}} = a$ (a constant Quantity *;) which is the very Equation found * Art. 406. in *Prob. 1.* belonging to a Circle.

If the Solid be supposed given, and the Superficies a *Minimum*, we shall come at the very same Conclusion:

For, $y^2 \dot{x}$ and $y \times \overline{x\dot{x} + y\dot{y}}^{\frac{1}{2}}$ (which are respectively as their Fluxions) being compared with $y^m \dot{u}$ and $\frac{y^r \times \overline{uu \pm yy}^n}{y^{2n-1}}$

we have $m = 2$, $\dot{u} = \dot{x}$, $r = 1$, and $n = \frac{1}{2}$; and there-

fore $\frac{\dot{x}}{y \sqrt{x^2 + y^2}}$ equal to a constant Quantity: Which

being denoted by $\frac{1}{a}$ (so that the Terms may be ho-

mologous) there comes out $a\dot{x} = y \sqrt{x^2 + y^2}$; whence $2ax - x^2 = y^2$ (as before.)

PROB. IV.

413. To determine the Curve HFB, from whose Revolution a Solid BK shall be generated; which, moving forward, in a Medium, in the Direction of its Axis DA, will be less resisted than any other Solid of the same given Length DA and Base BC.

If $AE = x$, $EF = y$, $Fp = \dot{x}$ &c. it is evident, from the Principles of Mechanics, that the resisting Force of a Particle of the Medium at F (being as the Square of the Sine of the Angle of Inclination pFq) will be truly

represented by $\frac{y\dot{y}}{x\dot{x} + y\dot{y}} \left(\frac{pq^2}{Fq^2} \right)$. Moreover, since

stance, equal to each other.) But, what a , itself, ought to be, must be determined from the given Values of AD and BD, and the Resolution of the foresaid Equation.

P R O B. V.

414. To determine the Solid of the least Resistance, supposing the Area of the generating Plane AHBD, and its greatest Ordinate DB to be given; (see the preceding Figure.)

Since (by the last Article) the Fluxion of the Resistance is expressed by $\frac{y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-1}}{\dot{y}^{-3}}$, and that of the

Area AEFH by $y\dot{x}$, it is plain (from the premised Theorem *) that $\frac{\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-2}}{\dot{y}^{-3}}$ is a constant Quantity. * Art. 406.

Whence, $\frac{\dot{y}^3 \dot{x}}{\overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^2}$, or its Equal $\frac{\overline{pq}^3 \times Fp}{qF^4}$, being

every where the same, the Angle pFq must also be invariable; and consequently HFB a Right-line. Therefore the Solid of the least Resistance is (in this Case) either a whole Cone, or the Frustrum of a, greater, Cone. But it is easy to shew, that, when the Area of the generating Plane AB is given so small, that the Angle B may be taken equal to the Half of a Right-angle; I say, it is demonstrable, in this Case, that the Frustrum so arising will be less resisted than a whole Cone, or any other Frustrum, whereof the Base and the Area of the generating Plane are the same.

In like manner the Solid of least Resistance, when its Bulk and greatest Diameter are given, may be determined: The Equation of the generating Curve being

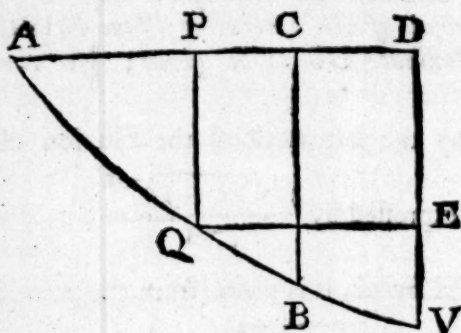
$$\frac{y^{-1} \dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-2}}{\dot{y}^{-3}} = \frac{1}{a}, \text{ or } a\dot{x}\dot{y}^3 = y \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^2 :$$

Whereof the Solution is given in Art. 264.

P R O B.

P R O B. VI.

415. To determine the Line, along which a Body, by its own Gravity, will, descend, from one given Point A to another B, in the shortest Time possible.



Let AD be parallel, and BC perpendicular, to the Horizon, intersecting each other in C; and let QP be any Ordinate to the Curve parallel to BC: Then (calling AP, x ; PQ, y &c.) the Celerity at Q will be expressed by $y^{\frac{1}{2}}$; also the Fluxion of the Time of Descent thro'

Art. 204. AQ will be truly defined by $\frac{\dot{z}}{y^{\frac{1}{2}}}$ *, or its Equal $y^{-\frac{1}{2}}$

$\times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{1}{2}}$. Here, therefore, the Fluent of $y^{-\frac{1}{2}} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{1}{2}}$ is to be a Minimum, when that of $\dot{x}$ arrives to

† Art. 405. to a given Value (AC). Whence, by the Theorem †,

$y^{-\frac{1}{2}} \dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-\frac{1}{2}}$ must be = a constant Quantity: Which (to have the Terms homologous) let be denoted by $a^{-\frac{1}{2}}$ (or $\frac{1}{\sqrt{a}}$). Then $a^{\frac{1}{2}} \dot{x} = y^{\frac{1}{2}} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{\frac{1}{2}}$;

whence $\dot{x} = \frac{y^{\frac{1}{2}} \dot{y}}{\sqrt{a - y}} = \frac{\dot{y}y}{\sqrt{ay - yy}}$; $\dot{z} = (\sqrt{\dot{x}^2 + \dot{y}^2})$
=

$$= \frac{a^{\frac{1}{2}}y}{\sqrt{a-y}}; \text{ and consequently } z = 2a - 2a^{\frac{1}{2}}\sqrt{a-y}.$$

Therefore, when $y = a$, z is $= 2a$; which two corresponding Values let be denoted by DV and AV; and let QE, parallel to AD, meet DV in E; then VE (VD — ED) being $= a - y$, and VQ (AV — AQ)

$$= 2a^{\frac{1}{2}}\sqrt{a-y}, \text{ it follows that}$$

$$VD(a) : VE(a-y) :: VA^2(4a^2) : VQ^2(4a \times a - y)$$

Which is one of the most remarkable Properties of the Cycloid; the Curve which, therefore, answers the Conditions of the Problem.

If the Celerity be supposed as any Function (S) of the Quantity y , the Problem will be resolved in the same manner: The Equation of the Curve being

$$\frac{\int \sqrt{xx + yy}^{-\frac{1}{2}}}{S} = \frac{1}{a} *.$$

• Art. 402.

P R O B. VII.

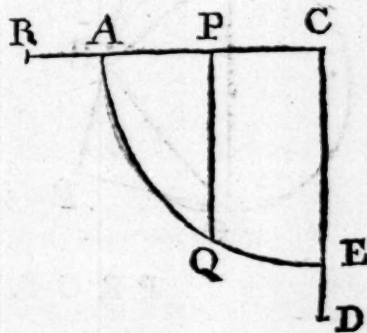
416. To find the Nature of the Curve AQE, along which a heavy Body must descend from an horizontal Line RC to a vertical Line CD, so that the Area CAE may be given, and the Time of the Descent a Minimum.

If the Ordinate PQ (parallel to CD) be called y , and the Velocity at Q be denoted by

$$y^n; \text{ it is evident that the Fluent of } y^{-n} \times \sqrt{xx + yy}^{\frac{1}{2}} (= \frac{z}{y^n} +)$$

must be a Minimum when that of y^n amounts to a given Value.

Therefore



† Art. 204.

492 *Of Problems De Maximis & Minimis*

Therefore (by the Theorem already mention'd so often) we have $y^{-n-1}\dot{x} \times \overline{\dot{x}\dot{x} + \dot{y}\dot{y}}^{-\frac{1}{2}} = a^{-n-1}$; and

consequently $\dot{x} = \frac{y^{n+1}\dot{y}}{\sqrt{a^{2n+2} - y^{2n+2}}}$; which, by writing $\frac{1}{2}$ instead of n , becomes $\dot{x} = \frac{y^{\frac{3}{2}}\dot{y}}{\sqrt{a^3 - y^3}}$: Whence

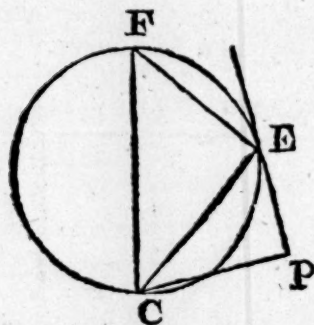
x will be known. But, if the Celerity was to be every where uniform, then (n being $= 0$) we should have

$$\dot{x} = \frac{y\dot{y}}{\sqrt{a^2 - y^2}}; \text{ and therefore } x = a - \sqrt{a^2 - y^2}:$$

Which answers to a Circle.

LEMMA.

417. *If, upon a Tangent EP, from any Point C in the Circumference of a Circle FEC, a Perpendicular CP be let fall, the Chord (CE) joining that Point and the Point of Contact, will be a Mean-Proportional between the said Perpendicular CP and the Diameter CF of the Circle.*



For, the Angles P and CEF being both Right; and also $\angle CEP = \angle F$, the Triangles CPE and CEF are similar: And therefore $CP : CE :: CE : CF$.

Q. E. D.

P R O B. VIII.

418. *In the mixt-lin'd Triangle ACB, the Lengths of all the Sides (whereof CA and CB are Right-lines) are supposed given; 'tis required to find the Nature of the Curve-side AEB, so that the Area may be a Maximum.*

Put

duced to meet it in R and Q, it follows that the whole Action upon the Body, in describing the Arch AE, will be proportional to the corresponding Part RQ of the said Tangent. For, if Ce be, also, produced to meet FR in q, and EF be drawn, it is plain that the Triangles CEF and CFQ, as also CEe and CqQ, are similar: Whence it will be, CE (y) : CF (a) :: CF (a)

$$: CQ \text{ (or } Cq) = \frac{aa}{y}; \text{ and CE (y) : Ee (z) :: Cq (}\frac{aa}{y}\text{)}$$

$$: Qq = \frac{aa\dot{z}}{yy} : \text{Which (a being constant) is as } \left(\frac{\dot{z}}{yy}\right)$$

the Force that acts upon the Body in describing Ee (z). And, as this every where holds, the whole Action in describing AE must therefore be proportional to RQ. Which Force (it is easy to prove) will be to that exerted on the Body in moving thro' the Chord AE, as the Chord to the Arch.

P R O B. X.

421. To determine the Path in which a Body may move from one given Point A to another B, in the shortest Time possible; supposing the Velocity to be, every where, proportional to any Power (y^p) of the Distance from a given Center C. (See the last Figure.)

Here every thing will remain as in the preceding Problem; only y^{-p} must be wrote instead of y⁻².

Therefore we have y^{-p+1} × u × $\sqrt{uu + yy}$ ^{-1/2} = a constant Quantity: Which Quantity (to have the Terms

homologous) let be denoted by $\frac{b}{a^p}$; then, by Reduction,

$$\frac{by^{p-1}}{a^p} = \frac{u}{\sqrt{uu + yy}} \left(= \frac{Ed}{Ee} \right) = \frac{CP}{CE} = \frac{CP}{y} :$$

And consequently CP = $\frac{by^p}{a^p}$. Hence, if p=0, or the

496 Of Problems De Maximis & Minimis

Velocity be constant ; then CP being every where $= b$, the Body must, in this Case, describe a Right-line.

But, if $p = 1$, then CP being $= \frac{by}{a}$; the Curve will

- Art. 74. be a logarithmic Spiral, whose Center is C* : Except in that particular Case, where $CA = CB$, when it degenerates to a Circle.

Lastly, if $p=2$, the Curve will be a Circle (by the preceding Lemma) whose Diameter is $\frac{aa}{b}$, and whose Periphery passes through the given Point C.

After the same manner, the Value of CP (upon which the Nature of the Curve depends) may be determined, when the Velocity is expounded by any given Function (S) of the Distance (y) from the Center of

- † Art. 407. Force : And (by writing S in the room of $y^p + \&c.$)

will come out $CP = \frac{bS}{c}$; where b and c represent constant Quantities.

When the Velocity is That which the Body may acquire, in descending thro' BE, by a centripetal Force expressed by y^p , then the Value of S (the Measure of

- ‡ Art. 221. that Velocity) being interpreted by $\sqrt{d^{p+1} - y^{p+1}}$ ‡ and 206,

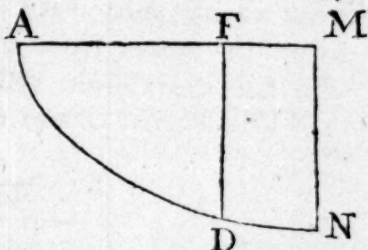
(where $CB=d$) we therefore have $CP = \frac{b\sqrt{d^{p+1} - y^{p+1}}}{c}$

for the Equation of the Curve of the swiftest Descent, according to this last Hypothesis of a centripetal Force varying as any Power p of the Distance.

422. Besides the Problems already resolved in this Section, there are others of the same Nature which are confined to more particular Restrictions, and require a different Method of Solution.

Thus,

Thus, if $\mathcal{Q}$, R and S be supposed to denote any given Powers, or Functions, of the Ordinate (y) of a Curve ANM, and the Nature of the Curve be required, so that, when the Fluent of $\mathcal{Q}\dot{z}$ becomes equal to a given Quantity, the Fluent of $R\dot{z}$, may also become equal to another



given Quantity, and That of $S\dot{z}$, a *Maximum* or *Minimum*: Then, because there is, in this Case, a second Equation, or new Condition, beyond what is to be met with in any of the foregoing Problems, the Method of Solution hitherto explained, will, therefore, be insufficient. But, by a Process similar to that whereby the said Method was demonstrated (assuming, here, three Expressions, and three indeterminate Quantities, instead of two *) a general Answer to this Problem (under all its Restrictions) will be obtained: And is exhibited by the Equation, * Art. 407.

$\frac{\dot{z}}{x} = \frac{pR \pm qS}{\mathcal{Q}}$; wherein p and q denote constant Quantities.

423. Though it seems unnecessary to put down the Invention of this Equation, after what has been hinted above, yet it may not be improper to observe, by way of Corollary, that, if $\mathcal{Q}=1$, $R=1$, and $S=y^n$, the

Equation will then become $\frac{\dot{z}}{x} = p \pm q^n$; expressing

the Nature of the Curve, when, the whole Abscissa (AM) and corresponding Arch (AN) being both given Quantities, the Fluent of $y^n \dot{z}$ is a *Maximum* or *Minimum*, according as the Value of n is positive or negative: In both which Cases, it is very easy to perceive, that the Curve must

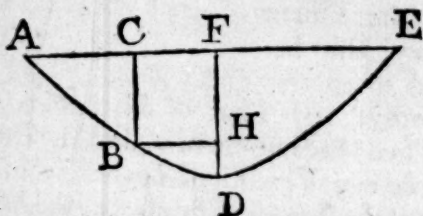
be concave to AM, and that the Value of $\frac{\dot{z}}{x}$, or its

K k

Equal

Equal $p \pm qy^n$, must, therefore, decrease as y increases; whence we may infer that the Sign of qy^n must be negative in the former Case, and positive in the latter.

Ex. Let the Curve ABDE, be the *Catenaria*; form'd by a slender Chain, or perfectly flexible Cord,



suspended by its two Extremes in the horizontal Line AE: Then, since its Center of Gravity must be the lowest possible, the Fluent of $y\dot{z}$, when $AC=AE$, must
 † Art. 173. therefore be a *Maximum*†: Whence (n being here $=1$)

our Equation $\left(\frac{\dot{z}}{\dot{x}} = p \pm qy^n\right)$ becomes $\frac{\dot{z}}{\dot{x}} = p - qy$.

But, in order to reduce it to a more convenient Form, let the Distance (DF) of the lowest Point of the Curve from the horizontal-Line AE be put $= b$; then, when y (BC) becomes $= b$, $\dot{x}$ will be $= \dot{z}$; and therefore the Equation, in that Circumstance, is $1 = p$

$- qb$; whence $p = 1 + qb$, and consequently $\frac{\dot{z}}{\dot{x}} =$

$1 + qb - qy = 1 + q \times \overline{b-y}$: Which, by putting

$b - y$ (DH) $= s$ and $a = \frac{1}{q}$ is reduced to $\frac{\dot{z}}{\dot{x}} = 1$

$+ \frac{s}{a}$: From whence $a^2 \dot{z}^2 (= \overline{a+s}^2 \times \dot{x}^2) = \overline{a+s}^2$

$\times \dot{z}^2 - s^2$; and consequently $BD = \sqrt{2as + ss}$.

For another Example (wherein the Exponent n will be negative) let the required Curve be That along which

which a Body may descend, by its own Gravity, from one given Point A to another B, in less Time than thro' any other Line of the same Length. In

which Case, the Fluent of $\dot{z}y^{-\frac{1}{2}}$ being a *Minimum*, when x and z become equal to given Quantities, our Equation (by writing $-\frac{1}{2}$ for n) will here become

$\frac{\dot{z}}{x} = p + qy^{-\frac{1}{2}}$: From whence exterminating x , or $\dot{z}$, by means of the Equation $\dot{x}^2 + y^2 = \dot{z}^2$, the Fluent may also be determined.

SECTION XI.

The Resolution of Problems of various Kinds.

PROB. I.

424. **A**NY hyperbolical Logarithm (y) being given, 'tis proposed to find the natural Number answering thereto.

If the Number sought be denoted by $1+x$, we shall

(by Art. 126.) have $y = \frac{\dot{x}}{1+x}$, or $y + xy - \dot{x} = 0$.

Let $Ay + By^2 + Cy^3 \&c. = x$; then $Ay + 2Byy' + 3Cy^2y' \&c. = \dot{x}$, and our Equation will become

$$\left. \begin{aligned} y + Ay' + By^2y' + Cy^3y' \&c. \\ - Ay - 2Byy' - 3Cy^2y' - 4Dy^3y' \&c. \end{aligned} \right\} = 0.$$

Whence, by comparing the homologous Terms, we

get $A=1$, $B = \frac{A}{2} = \frac{1}{2}$, $C = \frac{B}{3} = \frac{1}{2 \cdot 3}$, $D =$

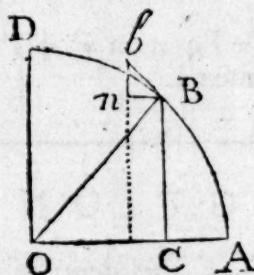
$\frac{C}{4} = \frac{1}{2 \cdot 3 \cdot 4} \&c.$ Therefore $1+y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} +$

$\frac{y^4}{2 \cdot 3 \cdot 4} + \frac{y^5}{2 \cdot 3 \cdot 4 \cdot 5} \&c.$ is $(= 1+x)$ the Number sought.

PROB. II.

425. The Radius AO and any Arch AB of a Circle ABD being given; to find the Sine BC, and Co-sine OC of that Arch.

Let AO (BO) = r , AB = z , AC = x , BC = y ,



$Bb = \dot{z}$, $Bn = \dot{x}$, and $bn = y$: Because of the similar Triangles OBC and Bnb, it will be

$$OB (r) : BC (y) :: Bb (\dot{z}) : Bn (\dot{x})$$

$$\text{And } OB (r) : OC (r-x) :: Bb (\dot{z}) : bn (y)$$

From which we have

$$yz = r\dot{x}$$

$$\text{And } ry = r\dot{z} - x\dot{z}.$$

$$\text{Let } x = Az + Bz^2 + Cz^3 + Dz^4 + Ez^5 \text{ \&c.}$$

$$\text{And } y = az + bz^2 + cz^3 + dz^4 + ez^5 \text{ \&c.}$$

Then, by Substitution and Transposition, our two Equations will become

$$\begin{aligned} & * + az\dot{z} + bz^2\dot{z} + cz^3\dot{z} + dz^4\dot{z} \text{ \&c.} \\ & -rA\dot{z} - 2rBz\dot{z} - 3rCz^2\dot{z} - 4rDz^3\dot{z} - 5rEz^4\dot{z} \text{ \&c.} \end{aligned} \quad \left. \vphantom{\begin{aligned} & * + az\dot{z} + bz^2\dot{z} + cz^3\dot{z} + dz^4\dot{z} \text{ \&c.} \\ & -rA\dot{z} - 2rBz\dot{z} - 3rCz^2\dot{z} - 4rDz^3\dot{z} - 5rEz^4\dot{z} \text{ \&c.} \end{aligned}} \right\} \parallel 0$$

And

$$\begin{aligned} & raz\dot{z} + 2rbz\dot{z} + 3rcz^2\dot{z} + 4rdz^3\dot{z} + 5rez^4\dot{z} \text{ \&c.} \\ & -r\dot{z} + Az\dot{z} + Bz^2\dot{z} + Cz^3\dot{z} + Dz^4\dot{z} \text{ \&c.} \end{aligned} \quad \left. \vphantom{\begin{aligned} & raz\dot{z} + 2rbz\dot{z} + 3rcz^2\dot{z} + 4rdz^3\dot{z} + 5rez^4\dot{z} \text{ \&c.} \\ & -r\dot{z} + Az\dot{z} + Bz^2\dot{z} + Cz^3\dot{z} + Dz^4\dot{z} \text{ \&c.} \end{aligned}} \right\} = 0.$$

From which, by equating the homologous Terms, we get $A=0$, $a=2rB$, $b=3rC$, $c=4rD$, $d=5rE$ \&c.

$$\text{Also } a=1, \quad b = -\frac{A}{2r}, \quad c = -\frac{B}{3r}, \quad d = -\frac{C}{4r} \text{ \&c.}$$

There-

Therefore $2rB = 1$, $3rC = -\frac{A}{2r}$, $4rD = -\frac{B}{3r}$,

$5rE = -\frac{C}{4r}$, &c. and consequently $B = \frac{1}{2r}$, $C = 0$,

$D = -\frac{B}{3 \cdot 4r^2} = -\frac{1}{2 \cdot 3 \cdot 4r^3}$, $E = 0$, $F = -$

$\frac{D}{5 \cdot 6r^2} = \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot r^5}$ &c.

Whence, also $b (= 3rC) = 0$, $c (= 4rD) = -\frac{1}{2 \cdot 3r^2}$ &c. &c.

Hence it is evident that $y (= az + bz^2 + cz^3 \text{ \&c.})$

$= z - \frac{z^3}{2 \cdot 3r^2} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5r^5} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7r^6}$

$+ \text{\&c.}$ And that $x (= Az + Bz^2 + Cz^3 \text{ \&c.}) = \frac{z^2}{2r} -$

$\frac{z^4}{2 \cdot 3 \cdot 4r^3} + \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6r^5} - \text{\&c.} \dagger$

PROB. III.

426. To find the Value of x , when x^x is a Minimum.

The Logarithm of x^x is $= x \times l. x$; whose Fluxion $\dot{x} \times l. x + x \text{ being } = 0$, we have $l : x = -1$. But (by Prob. 1.) the Number whose hyp. Log. is y will

be $1 + y + \frac{y^2}{2} + \frac{y^3}{2 \cdot 3} + \frac{y^4}{2 \cdot 3 \cdot 4} \text{ \&c.}$ Therefore, by

writing -1 instead of y , we have $x = 1 - 1 +$

$\dagger$ The Substance of this Solution (being the most neat and artful I have seen to that useful Problem) I had from a Letter sign'd — Needler; which was put into my Hands by a Friend, who receiv'd it from the late Dr. Halley, to whom it was wrote.

$$\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \mathcal{E}c. = 0,367878 \mathcal{E}c.$$

PROB. IV.

427. To divide a given Number (a) so that the continual Product of all its Parts may be a Maximum.

It is evident (from Art. 23.) that all the Parts must be equal: If, therefore, any one of them be denoted by x , their Number will be $\frac{a}{x}$, and we shall have

$\frac{a}{x}$ a Maximum: And therefore its Logarithm $\frac{a}{x} \times$

$L.x$ a Maximum also: And its Fluxion $-\frac{a\dot{x}}{x^2} \times L.x$

* Art. 22. and 126. $-\frac{a\dot{x}}{x^2} = 0$ *: Whence $H-L.x = 1$, and consequently

$$\dagger \text{ Art. 424. } x = 1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} \mathcal{E}c. = 2.71828$$

$\mathcal{E}c.$ Therefore the next inferior, or superior, Number to 2,71828 $\mathcal{E}c.$ that will exactly measure the given Number a , is the required Value of each Part:

Thus, let $a = 10$; then because $\frac{10}{2.71828 \mathcal{E}c.} = 4$

nearly, the Number of Parts, in this Case, will be 4,

and the Value of each $= \frac{10}{4} = 2.5$.

PROB. V.

428. To divide a given Angle AOB into two Parts AOC and BOC, so that the Product of any given Powers, $AP^n \times BQ^m$, of their Sines AP and BQ may be a Maximum.

Let

Let AP, produced, cut the Radius OB in D, and the Arch AB in F; likewise let FE and AL be perpendicular to OB, and join O, F: Putting $AO=r$, $AP=x$ and $BQ=y$. Then, because $x^n y^m$ is to be a *Maximum*, we have $nx^{n-1}\dot{x} \times y^m + x^n \times m y^{m-1}\dot{y} = 0$; and consequently $ny\dot{x} = -mxy\dot{y}$.

Moreover, since the Fluxion

of the Arch AC is $= \frac{r\dot{x}}{\sqrt{r^2-x^2}}$

and that of BC $= \frac{ry\dot{y}}{\sqrt{r^2-y^2}}$

(*Art. 142.*) we also have

$$\frac{ry\dot{y}}{\sqrt{r^2-y^2}} + \frac{r\dot{x}}{\sqrt{r^2-x^2}} = 0,$$

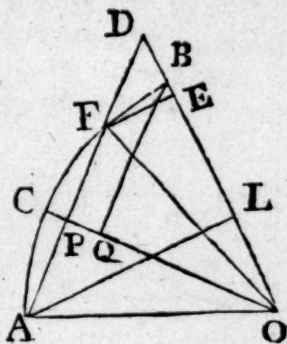
or $\frac{\dot{y}}{\sqrt{r^2-y^2}} = \frac{-\dot{x}}{\sqrt{r^2-x^2}}$; which multiply'd by the

former Equation, &c. gives $\frac{ny}{\sqrt{r^2-y^2}} = \frac{mx}{\sqrt{r^2-x^2}}$,

or $n \times \frac{y\sqrt{r^2-x^2}}{\sqrt{r^2-y^2}} = mx$: Whence, because OQ

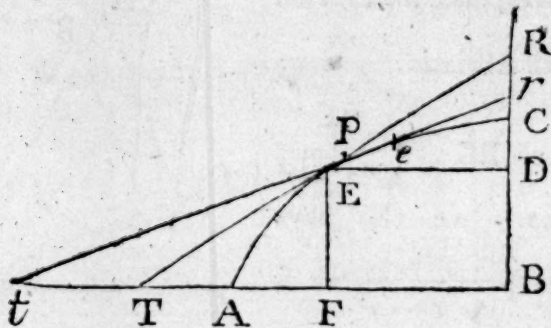
$(\sqrt{r^2-y^2}) : QB (y) :: OP (\sqrt{r^2-x^2}) : PD = \frac{y\sqrt{r^2-x^2}}{\sqrt{r^2-y^2}}$, we have $n \times PD (=mx) = m \times AP$;

and therefore $PD : AP :: m : n$; whence (by Composition and Division) $AD : DF :: m+n : m-n$: But (*by sim. Triang.*) $AD : DF :: AL : FE$; consequently $m+n : m-n :: AL : FE$; that is, as the Sum of the Indices of the two proposed Powers is to their Difference, so the Sine of the whole given Angle to the Sine of the Difference of its two, required, Parts. This Proportion is given in Words, at length, because it will be found of frequent Use in the Solution of mechanical Problems.



PROB. VI.

429. To shew that the least Triangle that can be described about, and the greatest Parallelogram in, a given Curve ABC, concave to its Axis, will be when the Subtangent FT is equal to the Base BF of the Parallelogram, or half the Base BT of the Triangle.



It appears from *Art. 25.* and is demonstrable by common Geometry, that the greatest Parallelogram that can be inscrib'd in the Triangle BTR (supposing the Position of TR to remain the same) will be *that* whose Base BF is half the Base BT of the Triangle: Therefore, as a greater Figure cannot possibly be inscribed in the Curve BAC than in the Triangle BTR circumscribing it, the greatest Parallelogram that can be inscribed, either, in the Triangle or the Curve, must be That above specified.

But now, to make it also appear that the Triangle BTR is a *Minimum* when $FT = BT$; let *Btr* be any other circumscribing Triangle, and let the two Tangents TER and *ter* intersect each other in P. Then, ER being $= ET$, it is plain that RP is less than PT, and Pr (less than PR less than PT) less than Pt: Therefore, the Sides PR and Pr of the Triangle RPr being less than the Sides, PT and Pt of the Triangle TPt, and the opposite Angles RPr and TPt equal to each other, it follows that the Triangle RPr is less than TPt; and consequently, by adding the Trapezium BTPr to both, it appears that BTR is less than Btr.

Cc.

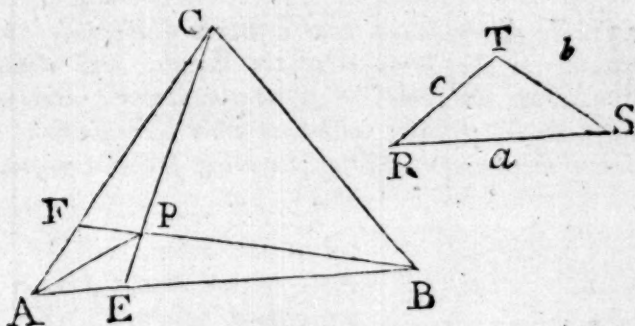
COROLLARY.

430. Hence the greatest inscribed Parallelogram is half the least circumscribing Triangle.

In the same Way it may be proved, that the greatest inscrib'd Cylinder, and the least circumscribing Cone, in, and about, the Solid generated by Revolution of a given Curve, will be when the Sub-tangent is equal to twice the Altitude of the Cylinder, or $\frac{2}{3}$ of the Altitude of the Cone: And that the two Figures will be to each other in the Ratio of 4 to 9.

P R O B. VII.

431. Three Points A, B, C being given, to find the Position of a fourth Point P, so that, if Lines be drawn from thence to the three former, the Sum of the Products $a \times AP$, $b \times BP$, and $c \times CP$ (where a , b and c denote given Numbers) shall be a Minimum.



If CP and BP be produced to E and F, it will appear from Art. 35. and 36. that the Sine of BPE must be to that of APE, as a to b ; and the Sine of CPF (BPE) to that of APF, as a to c . Therefore, the Sines of the three Angles BPE, APE, and APF (which Angles, taken all together, make two Right-ones) being in the given Ratio of a , b and c , it follows, that, if a Triangle RST be constructed, whose Sides RS, ST and RT are in the said Ratio of a , b and c , the Angles T, R and S opposite thereto, will be respectively equal

to the fore-mention'd Angles BPE, APE and APF. From whence, all the Angles at the Point P being given, the Position of that Point is given by common Geometry.

But it is observable, that, when one of the three given Quantities a, b, c (suppose a) is equal to, or greater than, the Sum of the other two, a Triangle cannot then be formed whose Sides are proportional to the said Quantities : In that Case the Point P will fall in the Point (A) corresponding to the greatest Quantity (a). For, it is plain that $b \times AB$ is less than $b \times BP + b \times AP$; and that $c \times AC$ is less than $c \times CP + c \times AP$; whence, by adding the Less to the Less, and the Greater to the Greater, it also appears that $b \times AB + c \times AC$ must be less than $b \times BP + c \times CP + b + c \times AP$ less than $b \times BP + c \times CP + a \times AP$; because a (by Hypothesis) is equal to, or greater than, $b + c$.

P R O B. VIII.

432. To determine in what Latitude a Right-line perpendicular to the Surface of the Earth, and Another drawn, from the same Point, to the Center, make the greatest Angle, possible, with each other; the Ratio of the Axis and the Equatoreal Diameter being supposed given.



Let AE represent the Equatoreal Diameter, and SP the Axis of the Earth (taken as an oblate Spheroid) also let RO and RM represent the two Lines specified in the Problem, whereof let the latter (perpendicular to ARS) meet SP in M; and let RB be perpendicular to SP.

It is evident, from the Property of the Ellipsis, that $SP^2 : AE^2 :: BO : BM$. And (by Trigonometry) $BO : BM :: \text{Tang. BRO} : \text{Tang. BRM}$; whence, by Equality,

lity, $SP^2 : AE^2 :: \text{Tang. BRO} : \text{Tang. BRM}$; therefore, by Composition and Division, $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Tang. BRM} + \text{Tang. BRO} : \text{Tang. BRM} - \text{Tang. BRO}$. But, the Sum of the Tangents of any two Angles is to their Difference, as the Sine of the Sum of those Angles to the Sine of their Difference*; whence it follows that $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Sine. } \overline{\text{BRM} + \text{BRO}} : \text{Sine. } \overline{\text{BRM} - \text{BRO}}$ (ORM).

Now, since the Ratio of the two first Terms is constant, or in every Part of the Ellipsis the same, it is obvious that the Angle ORM, or its Sine, will be the greatest possible, when its Antecedent (the Sine of $\overline{\text{BRM} + \text{BRO}}$) is the greatest possible, that is when $\overline{\text{BRM} + \text{BRO}} = a$ Right-Angle and its Sine = Radius. Therefore, in the proposed Circumstance, when ORM is a Maximum, our last Proportion will become $AE^2 + SP^2 : AE^2 - SP^2 :: \text{Radius} : \text{Sine of ORM}$: And half the Angle, so found, added 45° , will give (BRM) the Complement of the required Latitude; because $\overline{\text{BRM} + \text{BRO}}$ (or $2\text{BRM} - \text{ORM}$) being $= 90^\circ$, it is evident that $2\text{BRM} = 90^\circ + \text{ORM}$, and consequently $\text{BRM} = 45^\circ + \frac{1}{2} \text{ORM}$.

P R O B. IX.

433. Of all the Semi-cubical Parabolas, to determine that, whereof, the Length of the Curve being given, the Area shall be a Maximum.

The general Equation is $ax^2 = y^3$: Moreover, the

Area is universally $= \frac{3y^{\frac{3}{2}}}{5a^{\frac{1}{2}}}$, and the Length of the Curve

$$= \frac{(4a + 9y)^{\frac{3}{2}}}{27a^{\frac{1}{2}}} - \frac{8a}{27} \quad (\text{see Art. 137.}) \quad \text{Let the last of these}$$

be put $= c$, and, by ordering the Equation, you will get

* Vid. p. 56. of my Trigonometry.

get $y = \frac{a^{\frac{1}{3}} \times \overline{27c + 8a}^{\frac{2}{3}} - 4a}{9}$: Whence, $\frac{3y^{\frac{3}{2}}}{5a^{\frac{1}{2}}}$ (and

consequently $\frac{y}{a^{\frac{1}{5}}}$) being a *Maximum*, it is evident that

$\frac{a^{\frac{1}{3}} \times \overline{27c + 8a}^{\frac{2}{3}} - 4a}{a^{\frac{1}{5}}}$, or its Equal $a^{\frac{2}{15}} \times \overline{27c + 8a}^{\frac{2}{3}}$

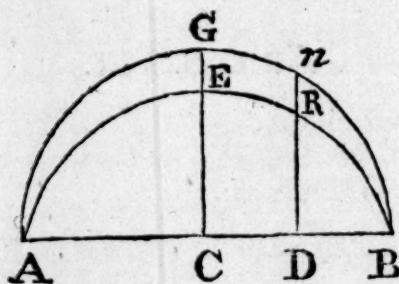
$- 4a^{\frac{4}{5}}$ must likewise be a *Maximum* : Which, put into Fluxions and reduced, gives $a = c \times \frac{9 + 3\sqrt{21}}{32}$:

Whence x and y will also be found.

PROB. X.

434. *To determine the Ratio of the Periphery of any given Ellipsis to that of its circumscribing Circle.*

Call the Semi-transverse Axis CB, a ; the Semi-conjugate CE, c ; any Ordinate DR, y ; and its Distance



CD from the Center, x : Then (by the Nature of the Curve) y being $= \frac{c}{a} \sqrt{aa - xx}$, we have $y' =$

$\frac{-cx}{a\sqrt{aa - xx}}$; and consequently $\dot{x} (\sqrt{\dot{x}^2 + y'^2}) =$
 $\frac{\dot{x} \sqrt{a^4 - a^2 - c^2} \times x^2}{a\sqrt{aa - xx}}$: Which, by making $d =$

$\frac{aa - cc}{aa}$ will be reduced to $z = \frac{x \sqrt{aa - dxx}}{\sqrt{aa - xx}} =$

$$\frac{ax}{\sqrt{aa - xx}} \times 1 - \frac{dx^2}{2a^2} - \frac{d^2x^4}{2 \cdot 4a^4} - \frac{3d^3x^6}{2 \cdot 4 \cdot 6a^6} \text{ \&c.}$$

(by throwing the Numerator into a Series) whereof the whole Fluent, when x becomes $= a$, will be z (ERB)

$$= A \times 1 - \frac{d}{2 \cdot 2} - \frac{3d^2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5d^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} -$$

$$\frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7d^4}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8} \text{ \&c. (by Art. 286.) where } A$$

denotes the Length of the Arch GnB, or $\frac{1}{4}$ of the Periphery of the circumscribing Circle.

Hence it follows that the Periphery of the Ellipsis is to that of its circumscribing Circle, as $1 - \frac{d}{2 \cdot 2} -$

$$- \frac{3d^2}{2 \cdot 2 \cdot 4 \cdot 4} - \frac{3 \cdot 3 \cdot 5d^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \text{ \&c. or as } 1 -$$

$$\frac{d}{2 \cdot 2} \times A + \frac{1 \cdot 3d}{4 \cdot 4} \times B + \frac{3 \cdot 5d}{6 \cdot 6} \times C + \frac{5 \cdot 7d}{8 \cdot 8} \times D$$

\&c. to Unity: Where A, B, C, D \&c. denote the preceding Terms, under their proper Signs.

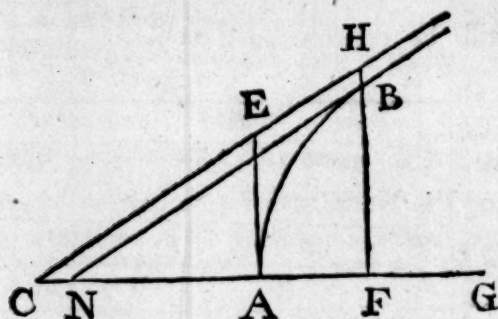
P R O B. XI.

435. To determine the Difference between the Length of the Arch of a Semi-hyperbola infinitely produced, and its Asymptote.

Call the Semi transverse Axis (AC) a ; the Semi-conjugate (or its Equal AE) b ; the Distance (CF) of any Ordinate from the Center, x ; the Ordinate itself, y ; and the Arch corresponding, z : Then, from the

Nature of the Curve we have $y = \frac{b \sqrt{x^2 - a^2}}{a}$; whence

$$y =$$



$$y = \frac{bx\dot{x}}{a\sqrt{x^2 - a^2}}; \text{ and consequently } z (= \sqrt{\dot{x}^2 + \dot{y}^2}) =$$

$$\frac{\dot{x} \sqrt{\frac{aaxx + bbxx}{aa} - a^2}}{\sqrt{xx - aa}} : \text{ Which, making } d^2 = \frac{a^2}{a^2 + b^2}$$

$$\left(= \frac{CA^2}{CE^2} \right) \text{ and } u = \frac{a}{x} \text{ will be transformed to } z =$$

$$- \frac{ai}{du^2} \times \frac{(1 - ddu)^{\frac{1}{2}}}{(1 - uu)^{\frac{1}{2}}}; \text{ whereof the upper Surd, ex-}$$

$$\text{panded, is } = 1 - \frac{d^2u^2}{2} - \frac{d^4u^4}{8} \text{ \&c. And therefore } z =$$

$$\frac{a}{d} \text{ into } \frac{-i}{u^2\sqrt{1-uu}} + \frac{d^2i}{2\sqrt{1-uu}} + \frac{d^4u^2i}{8\sqrt{1-uu}} +$$

$$\frac{3d^6u^4i}{8.6\sqrt{1-uu}} + \frac{3.5d^8u^6i}{8.6.8\sqrt{1-uu}} \text{ \&c. Now the}$$

$$\text{Fluent of the first Term hereof, } \frac{a}{d} \text{ into } \frac{-i}{u^2\sqrt{1-uu}}$$

$$\left(= \frac{x\dot{x}}{d\sqrt{x^2 - a^2}} \right) \text{ is universally expreffed by}$$

$$\frac{\sqrt{x^2 - a^2}}{d}, \text{ or its Equal } \frac{BF \times CE}{AE} : \text{ Which, if BN}$$

be parallel to the Asymptote EC, will (because AE : CE

CE :: BF : BN) be also truly represented by BN : And this Line BN, when x or z becomes infinite, will coincide with the Asymptote. Therefore the Fluent of the remaining Terms is the Difference sought : Which Fluent, when $u = 1$, or $y = 0$ (putting A for $\frac{1}{4}$ of the Periphery of the Circle whose Radius is Unity)

$$\text{will be} = aA \times \frac{d}{2} + \frac{d^3}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3d^5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} +$$

$$\frac{3 \cdot 3 \cdot 5 \cdot 5d^7}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7d^9}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10} \text{ \&c.}$$

(by Art. 286.) but $= 0$ when $u = 0$ (or y is infinite). Therefore the Excess of the Asymptote above the Curve is truly exhibited by the preceding Series. *Q. E. I.*

If a be taken $= 1$, and $b = 0$, then d will become $= 1$: And therefore, the Curve in this Case falling

$$\text{into its Axis AG, we have } A \times \frac{1}{2} + \frac{1}{2 \cdot 2 \cdot 4} +$$

$$\frac{3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} + \frac{3 \cdot 3 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8} \text{ \&c.} = CA,$$

or Unity. Whence it appears that the Sum of the Se-

$$\text{ries } \frac{1}{2} + \frac{1}{2 \cdot 2 \cdot 4} + \frac{3 \cdot 3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6} \text{ is the Reciprocal}$$

of $\frac{1}{4}$ of the Periphery of the Circle whose Radius is Unity. And, from the Problem preceding the last, it will likewise appear, that the Sum of the Series $1 -$

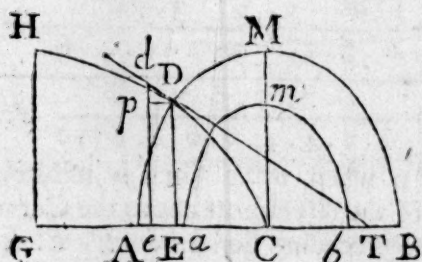
$$\frac{1}{2 \cdot 2} - \frac{3}{2 \cdot 2 \cdot 4 \cdot 4} + \frac{3 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6} \text{ \&c. will be}$$

denoted by the same Quantity ; and consequently that these two Serieses are equal to each other. From the Addition and Subtraction of which and their Multiples, various other Serieses may be produced, whose Sums are explicable by means of the Periphery of a Circle.

P R O B.

P R O B. XII.

436. To determine the Nature of the Curve CDH, which will intersect any Number of similar and concentric Ellipses AMB, amb &c. at Right-Angles.



Let the Tangent DT, which is a Normal to the Ellipsis AMB, meet the Axis AB in T; and, supposing AC, CM, aC, Cm &c. to be the principal Semi-diameters of

their respective Ellipses, let the given Ratio of AC^2 to CM^2 (or of aC^2 to Cm^2 &c.) be that of 1 to n : Putting $CE = x$, $ED = y$, $Dp (Ee) = \dot{x}$, and $dp = \dot{y}$.

It is a known Property of the Ellipsis that $AC^2 : CM^2 :: CE : ET$; therefore $ET = nx$: Moreover $ET (nx) : Dp (\dot{x}) :: ED (y) : pd (\dot{y})$ by similar Triangles

whence $\frac{\dot{x}}{nx} = \frac{\dot{y}}{y}$, or $\frac{\dot{x}}{x} = \frac{ny}{y}$; whereof the Fluent

* Art 126. is $L : x - L : a = nL : y - nL : a$ * (where a denotes any constant Quantity at Pleasure.) Hence we also

have $L : \frac{x}{a} = n \times L : \frac{y}{a} = L : \frac{y^n}{a^n}$, and consequently

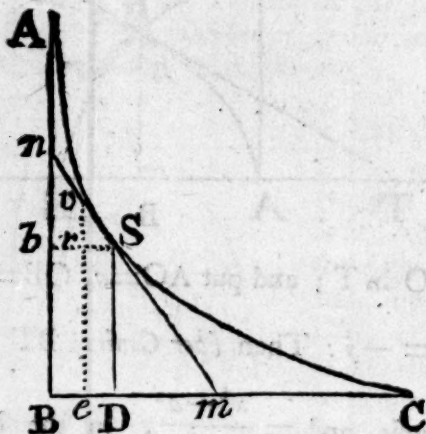
$$\frac{x}{a} = \frac{y^n}{a^n}, \text{ or } a^{n-1}x = y^n.$$

P R O B. XIII.

437. To find the Equation of a Curve ERD that will cut any Number of Ellipses, or Hyperboles, having the same Center O and Vertex A, at Right-Angles.

Let RT be a Tangent to any one of the proposed Conic Sections ARF, at the Intersection R, meeting the

$$\therefore y : \frac{y\dot{x}}{\dot{y}} = Dm, \text{ and } \dot{x} : \dot{y} :: a-x \text{ (Sb)} : \frac{a-x \times \dot{y}}{\dot{x}} =$$



bn : Whence Cm ($CD - Dm$) $= x - \frac{y\dot{x}}{\dot{y}}$, and Bn ($Bb + bn$) $= y + \frac{a-x \times \dot{y}}{\dot{x}}$: Which two last Values, because the Velocities of the Bodies are equal, must also be equal to each other, that is, $x - \frac{y\dot{x}}{\dot{y}} = y + \frac{a-x \times \dot{y}}{\dot{x}}$: Hence, by making $\dot{x}$ constant, and taking the Fluxion of the whole Equation, we get $\dot{x} - \frac{y\dot{x}\dot{y} + y\dot{x}\ddot{y}}{\dot{y}^2} = \dot{y} - \frac{\dot{x}\dot{y} + a-x \times \ddot{y}}{\dot{x}}$; or $\frac{a-x \times \ddot{y}}{\dot{x}} = \frac{y\dot{x}\ddot{y}}{\dot{y}^2}$; from which there arises $a-x \times \dot{y}^2 = y\dot{x}^2$, and $\frac{\dot{y}}{\sqrt{y}} = \frac{\dot{x}}{\sqrt{a-x}}$: Where, the Fluent on both Sides being taken, we have $2\sqrt{y} = 2\sqrt{a} - 2\sqrt{a-x}$, and consequently $x = 2\sqrt{ay} - y$: Which Equation pertains to the common Parabola.

Other-

Otherwise more universally, thus.

439. Put $Cm=v$ and $Bn=w$, and let these Quantities (instead of being equal) have any given Relation to each other. Then, since the absolute Celerity of m is expressed by $\dot{v}$, its angular Celerity, in a Direction perpendicular to Sm , by which the Line Sm tends to revolve about the Point of Contact S as a Center, will

be truly defined by $\frac{\text{Sine of } Emn}{\text{Radius}} \times \dot{v}$ (Art. 35.)

In the same manner the angular Celerity of n , about the Point S , will be defined by $\frac{\text{Sin. } Bnm}{\text{Rad.}} \times \dot{w}$. Now,

as these Celerities must be to each other as the Distances Sm and Sn from the Center S (or directly as the Radii) we have $Sm : Sn (:: DS : bn) :: \text{Sin. } Bnm \times \dot{v} : \text{Sin. } Bnm \times \dot{w}$; whence, because $\text{Sin. } Bnm : \text{Sin. } Bnm :: Bn (w) : Bm (a-v)$ we also have $DS : bn ::$

$w \times \dot{v} : a-v \times \dot{w}$: Therefore, by Composition, $DS : (DS + bn) w :: w \dot{v} : w \dot{v} + a-v \times \dot{w}$, and conse-

quently $DS = \frac{w^2 \dot{v}}{w \dot{v} + a-v \times \dot{w}}$: Whence $bn (w -$

$SD) = \frac{a-v \times w \dot{v}}{w \dot{v} + a-v \times \dot{w}}$; and $BD (= Sb = \frac{bn \times Bm}{Bn})$

$= \frac{a-v^2 \times \dot{v}}{w \dot{v} + a-v \times \dot{w}}$: From whence the Curve itself

will be given.

If v and w be taken equal to each other (as above)

then $SD (y)$ will become $= \frac{w^2}{a}$, and $BD = \frac{a-w^2}{a}$

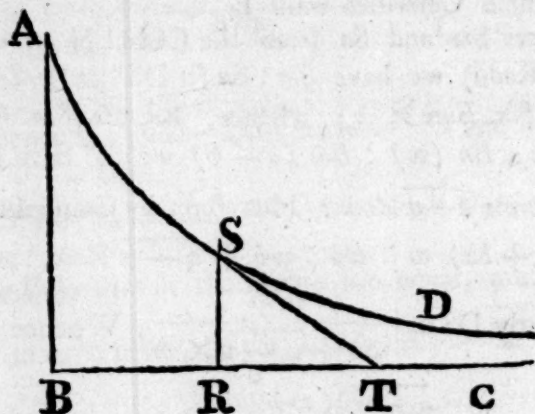
$= a - 2w + \frac{w^2}{a}$; in which last, if for w its Equal

$\sqrt{ay}$ be substituted, we shall have $BD = a - 2\sqrt{ay} + y$; and consequently $CD (a - BD) = 2\sqrt{ay} - y$, the very same as before.

P R O B. XV.

440. Supposing a Body T to proceed, uniformly, along a Right-line BC, and another Body S, in pursuit of the same, always directly towards it, with a Celerity which is to that of T, in any given Ratio, of 1 to n ; 'tis proposed to find the Equation of the Curve ASD described by the latter.

Let the Tangent AB, which makes Right-Angles with BC, be put $= a$, $BR = x$, $RS = y$, and $AS = z$:



Then the Subtangent RT being $= \frac{y\dot{x}}{\dot{y}}$, we have BT

$= x + \frac{y\dot{x}}{\dot{y}}$: Moreover, since the Distances BT and

AS gone over in the same time, are as the Celerities n and 1, we also have $BT (= n \times AS) = nz = x +$

$\frac{y\dot{x}}{\dot{y}}$: Whence, in Fluxions (making y constant) $\frac{-y\ddot{x}}{\dot{y}}$

$= n\ddot{z}$; and consequently $\frac{-ny}{y} \left(= \frac{\ddot{x}}{\ddot{z}} \right) = \frac{\ddot{x}}{\sqrt{y^2 + x^2}}$:

The

The Fluent of which (by Art. 126.) is $-n \times \text{Log. } y$

$$= \text{Log. } \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}} : \text{ But when } y=a, \dot{x} \text{ is } = 0,$$

and then the Equation becomes $-n \times \text{Log. } a = 0$; therefore the Fluent, duly corrected, is $n \times \text{Log. } a - n$

$$\times \text{Log. } y = \text{Log. } \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}, \text{ or } \text{Log. } \frac{a^n}{y^n} =$$

$$\text{Log. } \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}. \text{ Whence it is evident that } \frac{a^n}{y^n}$$

$$= \frac{\dot{x} + \sqrt{\dot{y}^2 + \dot{x}^2}}{\dot{y}}, \text{ and } \frac{a^n \dot{y}}{y^n} - \dot{x} = \sqrt{\dot{y}^2 + \dot{x}^2};$$

from which, by squaring both Sides, $2\dot{x}$ is found =

$$\frac{a^n \dot{y}}{y^n} - \frac{y^n \dot{y}}{a^n}; \text{ whose Fluent is } 2x = -\frac{a^n y^{1-n}}{1-n} +$$

$$\frac{a^{-n} y^{n+1}}{n+1}. \text{ But when } y=a, x \text{ is } = 0, \text{ and then,}$$

$$0 = -\frac{a}{1-n} + \frac{a}{n+1} = -\frac{2na}{1-nn}; \text{ therefore the}$$

$$\text{Fluent corrected is } 2x = -\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1} +$$

$$\frac{2na}{1-n^2}. \quad \mathcal{Q}. E. I.$$

Otherwise (without second Fluxions.)

441. Put $ST = P$ and $RT = \mathcal{Q}$. Then since the absolute Velocity of the Body S is denoted by Unity, that with which the Ordinate SR is carry'd towards the

Body T will be denoted by $\frac{\mathcal{Q}}{P} \times 1$ or $\frac{\mathcal{Q}}{P}$ (by Art. 35.)

which subtracted from n the Velocity of T , leaves $n - \frac{\mathcal{Q}}{P}$ for the relative Celerity with which T recedes from

R: After the same Manner, if from $\frac{Q}{P} \times n$ the Celerity of T in the Direction ST produced, there be taken (1) the Celerity of S in the same Direction, the Remainder, $\frac{nQ}{P} - 1$, will be the Celerity with which T recedes from S : Therefore, the Fluxions of Quantities being as the Celerities of their Increase, we have $n - \frac{Q}{P} : \frac{nQ}{P} - 1 :: \dot{Q} : \dot{P}$; and consequently $\overline{nQ - P} \times \dot{Q} = \overline{nP - Q} \times \dot{P}$. But, since the Quantities P and Q are concerned exactly alike, the Equation thus derived will, in all probability, become more simple, by substituting for their Sum and Difference: Let therefore $P + Q = s$, and $P - Q = v$, or, which is the same, let $P = \frac{s+v}{2}$, and $Q = \frac{s-v}{2}$:

Then, by Substitution, we shall have $\frac{ns - nv - s - v}{2}$
 $\times \frac{\dot{s} - \dot{v}}{2} = \frac{ns + nv - s + v}{2} \times \frac{\dot{s} + \dot{v}}{2}$; which contracted, &c. becomes $\overline{1+n} \times v \dot{s} = \overline{1-n} \times s \dot{v}$, or $\overline{1+n} \times \frac{\dot{s}}{s} = \overline{1-n} \times \frac{\dot{v}}{v}$; whose Fluent (corrected) is $\overline{1+n} \times \text{Log. } s = \overline{1-n} \times \text{Log. } v + 2n \times \text{Log. } a$, or $\text{Log. } s^{1+n} = \text{Log. } a^{2n} v^{1-n}$. Whence $s^{1+n} = a^{2n} v^{1-n}$, and consequently $s^{1+n} \times v^{1+n} = a^{2n} v^2$: But $sv (= \overline{ST+RT} \times \overline{ST-RT} = RS^2) = y^2$; therefore $s^{1+n} \times v^{1+n} = y^{2n+2} = a^{2n} v^2$, and $v = \frac{y^{n+1}}{a^n}$; whence $s \left(= \frac{y^2}{v} \right) = \frac{a^n}{y^{n-1}}$, $ST \left(\frac{s+v}{2} \right) \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}, RT \left(\frac{s-v}{2} \right)$

3
=

$$= \frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n}. \text{ But RS } (y) : \text{RT} \left(\frac{a^n}{2y^{n-1}} - \frac{y^{n+1}}{2a^n} \right)$$

$$:: y : \dot{x}; \text{ whence } 2\dot{x} = \frac{a^n \dot{y}}{y^n} - \frac{y^n \dot{y}}{a^n}, \text{ and } 2x = -$$

$$\frac{a^n y^{1-n}}{1-n} + \frac{a^{-n} y^{n+1}}{n+1} + \frac{2na}{1-nn}, \text{ the very same as before.}$$

COROLLARY.

442. If the Velocity of S be greater than that of T (or n be less than Unity) the two Bodies will concur when the latter has moved over a Distance expressed by

$$\frac{na}{1-n^2}; \text{ because, when } y \text{ becomes } = 0, 2x \text{ is barely } =$$

$$\frac{2na}{1-n^2}. \text{ But if the Velocity of } S \text{ be less than that of}$$

T , it is plain that S can never come up with T : But its

nearest Approach will be when $y = \frac{n-1}{n+1} \sqrt[n]{\frac{1}{2n}} \times a$: For,

$$\text{since ST is universally } = \frac{a^n}{2y^{n-1}} + \frac{y^{n+1}}{2a^n}, \text{ let the Flux-}$$

ion of this Expression be taken and put equal to Nothing; and y will be found as above exhibited.

If the Celerities of S and T , instead of being uniform, vary according to a given Law; then, denoting the former by A and the latter by B , the Equation of

$$\text{the Curve will be } \frac{\ddot{x}}{\sqrt{y^2 + \dot{x}^2}} = -\frac{B\dot{y}}{A\dot{y}}: \text{ And if the}$$

Fluent of $-\frac{B\dot{y}}{A\dot{y}}$ be explicable by a Logarithm, as $L. N$;

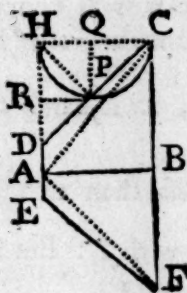
then, the Fluent of $\frac{\ddot{x}}{\sqrt{y^2 + \dot{x}^2}}$ being $L. \frac{y + \sqrt{y^2 + \dot{x}^2}}{y}$ * * Art. 126.

we shall have $N = \frac{y + \sqrt{y^2 + x^2}}{y}$; which, order'd,

gives $x = \frac{Ny}{2} - \frac{y}{2N}$: Whence x will be found.

P R O B. XVI.

443. To determine the Frustum CDEF of a Triangular-Prism, of a given Base CF and Altitude BA; which, moving in a Medium, in the Direction of its Length BA, shall be resisted the least possible.



Draw CH parallel to BA meeting ED, produced, in H: Moreover, let HP, PQ and PR be perpendicular to CD, CH and DH respectively.

Since the Number of resisting Particles acting upon DC is as DH, and the Force of each as

$\left(\frac{DR^2}{DP^2}\right)$ the Square of the Sine of

the Angle of Incidence DPR, the whole Resistance sustained by DC will therefore be expressed by $\frac{DH \times DR^2}{DP^2}$, or DR, which is equal to it (by

the Similarity of the Triangles DHP and DPR) Whence the Resistance upon ADC is truly expressed by AR (AD + DR) and is a *Minimum* when its Defect (PQ) below the given Quantity AH (or BC) is a *Maximum*; But PQ is a *Maximum* when CQ and HQ are equal; because, the Angle CPH being Right, a Semi-circle described upon CH will always pass thro' the Point P; and it is well known that the greatest Ordinate in a Semi-circle is That which divides the Diameter into two equal Parts.

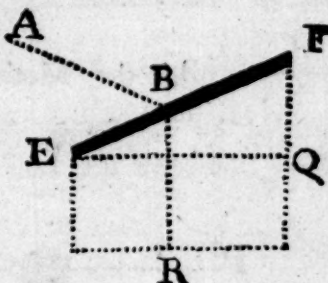
Hence the Angle DCH, when the Resistance upon ADC is a *Minimum*, will be just the Half of a Right-Angle, provided BC be given greater than BA; otherwise,

wife, the whole Prism CAF will be less resisted than any Fruustum CDEF of a greater Prism.

PROB. XVII.

444. To determine the Angle RBE which a Plane EBF must make with the Wind blowing in a given Direction RB, so that the Plane itself may be urged in another given Direction BA with the greatest Force possible.

It is known, from the Resolution of Forces, that the Force whereby the Plane EF is urged in the given Direction BA, by a Particle of Air, acting in the Direction RB, is directly as the Rectangle of the Sines of the Angles (ABE, RBE)



which the two given Directions make with the Plane: Therefore, since the Number of Particles acting on EF is as the Sine of RBE, it follows that the whole Force, or Effect, of the Wind, in the Direction BA, will be as $S. ABE \times Squ. S. RBE$; which being a *Maximum*, we have (by Prob. 5.) $3 : 1 :: \text{Sine of the whole given Angle RBA} : \text{Sine of } RBE - ABE$. Whence the Angles RBE and ABE are both given. *Q. E. I.*

COROLLARY.

445. If the Angle RBA be a Right one (which is the Case with regard to the Sails of a Windmill) then the Sine of $RBE - AEB$ being $= \frac{1}{3} = ,333$ &c. we shall have $RBE - AEB = 19^\circ : 28'$; and consequently

$$RBE \left(\frac{RBA + AEB}{2} \right) = 54^\circ : 44'.$$

PROB. XVIII.

446. If two Bodies A and B, joined by a String, be urged in opposite Directions, towards P and Q, by any given Forces F and f, uniformly continu'd; 'tis propos'd to find the Tension of the String, or the Force whereby the Bodies endeavour to recede from each other.

I

Since

Since $F - f$ is the absolute Force by which the two Bodies are, constantly, urged towards P, the *whole* Motion, generated in Both, in any Time T , will therefore be expressed by $\overline{F - f} \times T$: Whence, because both Bodies (by reason of the String) acquire the same Velocity, the Motion generated in A , alone, will be

$\frac{A}{A+B} \times \overline{F - f} \times T$, or that Part of the *Whole* defined by

$$\frac{A}{A+B}.$$

But the Motion of A , had it not been retarded by the String (or B) would have been $F \times T$; therefore the Loss of Motion, by the Action



upon the String, is $F \times T - \frac{A}{A+B} \times \overline{F - f} \times T$,

$= \frac{fA + FB}{A+B} \times T$: Which, divided by the Time T

(wherein that Loss or Effect is produced) gives $\frac{fA + FB}{A+B}$,

for the Tension of the Thread, or the Force sufficient to cause the said Loss of Motion.

The same otherwise.

447. Because the Force F , was it to act alone, would communicate, by means of the String, the same Velocity to B as to A , the Part therefore of the Force F employ'd upon B , by which the String is stretch'd, will

be $\frac{B}{A+B} \times F$, or $\frac{BF}{A+B}$: And, from the very same

Argument, if the Force f was to act alone, the Tension of the Thread would be $\frac{fA}{A+B}$: Therefore, when both

the

the Forces act together, the Tension will be $\frac{fA+BF}{A+B}$:

For it is very plain that, their acting both at the same time, no way influences their respective Effects on the Thread.
Q. E. I.

COROLLARY.

448. If the Forces F and f be respectively expounded by the Masses, or Weights, of the Bodies A and B ; the

Tension of the Thread will then become $\frac{2AB}{A+B}$.

Whence it appears that the Tension of a Thread sliding over a Pin or Pulley, by means of two unequal Weights A and B , suspended at the Ends thereof, is equal to

$\frac{2AB}{A+B}$: The Double whereof, or $\frac{4AB}{A+B}$, is the Weight which the Pin or Pulley sustains, while the Bodies are in Motion; because the Thread hangs double, or on both Sides the Pulley.

If several Bodies A, B, C, D &c. communicating by means of a String or Wire AF , be urged towards a Point P , in the Direction of the String or Wire, by any given Forces p, q, r, s &c. respectively, the Tension of the Part AB will be

$$= \frac{p \times B + C + D \text{ \&c.} - A \times q + r + s \text{ \&c.}}{A + B + C + D \text{ \&c.}};$$

of the Part BC

$$= \frac{p + q \times C + D + E \text{ \&c.} - A + B \times r + s + t \text{ \&c.}}{A + B + C + D \text{ \&c.}};$$



of CD

$$= \frac{p + q + r \times D + E + F - A + B + C \times s + t + v}{A + B + C + D \text{ \&c.}};$$

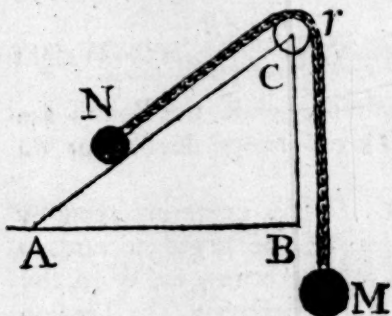
&c. &c.

All

All which easily follows from above ; and will answer also in those Cases where some of the Forces are supposed to act in the contrary Direction, if every such Force be considered as a negative Quantity.

P R O B. XIX.

449. Let it be required to raise a given Weight N , to a given Height BC , along an inclin'd Plane AC , by means of another given Weight M , connected to the former by a flexible Rope NrM , moving over a Pulley at C ; to find the Tension of the Rope ; also the Inclination and Length of the Plane, so that the Time of the whole Ascent may be the least possible.



It is well known that the Force by which N tends to descend along the Plane AC , or acts in opposition to M (supposing $BC=a$, and AC

$$=x) \text{ will be } \frac{aN}{x} :$$

$$\text{Therefore } M - \frac{aN}{x},$$

or $\frac{xM - aN}{x}$ is the efficacious Force, by which the

Bodies are accelerated : But it is likewise demonstrable that the Time of describing any Line by means of a Velocity uniformly accelerated, is in the subduplicate Ratio of the Length thereof, directly, and the subduplicate Ratio of the accelerating Force, inversely * : Whence it follows that the Time of describing AC will be

represented by $\frac{x}{\sqrt{xM - aN}}$: Whose Fluxion (or that of its Square) being made equal to Nothing, x will be found $= \frac{2aN}{M}$, or $M : 2N :: a : x$. Hence the

Time

* Art. 203.

Time of the Ascent will be the least possible, when the Sine of the Plane's Inclination is to the Radius, as the Power (M) is to twice the Weight (N) to be raised.

The Tension of the Rope will be determined from the last Problem, (by writing N for A , $\frac{aN}{x}$ for F , M

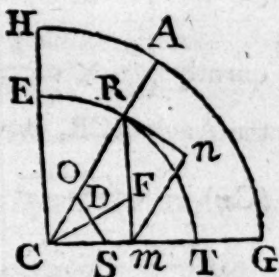
for B , and M for f) and comes out $= \frac{MN}{M+N} \times \frac{a+x}{x}$.

Q. E. I.

PROB. XX.

450. Let AC represent a Piece of Timber, moveable about a Center C , making any Angle ACG with the Plane of the Horizon CG ; to determine the Position of a Prop or Supporter OS , of a given Length, which shall sustain it with the greatest Facility, in any given Position; and also what Inclination AC will have to the Horizon when the least Force that can sustain it, is greater than the least Force in any other Position.

Let R be the Center of Gravity of the Beam AC , and let Rn , Rm and CD be perpendicular to AC , CG and OS respectively: Putting $SO=a$, $CR=r$, $Cm=x$, and the Weight of the Beam $=w$.



Then, by the Principles of Mechanics, we shall have, first,

as $Rm : Rn$, or as, $r : x :: w : \left(\frac{xw}{r}\right)$ the Force,

which acting at R , in the Direction Rn , is sufficient to sustain the Beam AC ; secondly, as $CO : CR (r) :: \frac{xw}{r}$

(the Quantity last found) : $\frac{xw}{CO}$, the Force able to support it, at O , in a perpendicular Direction; and, lastly,

lastly, as $CD : CO :: \frac{xw}{CO} : \frac{xw}{CD}$, the Force, or Weight, actually sustained by the given Prop SO. Which Force will therefore be the least possible when the Perpendicular CD is the greatest possible, let the Angle of Inclination GCA be what it will: But of all Triangles, having the same Base (OS) and vertical Angle (SCO) the Isosceles one is known to have the greatest Perpendicular: Therefore the Triangle CSO will be Isosceles, and the Angles S and O equal to each other, when the Weight sustain'd by the Prop OS is a *Minimum*.

But, now, to give a Solution to the latter Part of the Problem, or to find (supposing the Angles S and

O to be equal) when $\frac{x}{CD} \times w$ is a *Maximum*, let CD produced meet mR in F; and then, because of the similar Triangles CDS and CmF, we shall have $CD : x$

(Cm) :: SD ($\frac{1}{2}a$) : mF, or $\frac{x}{CD} = \frac{mF}{\frac{1}{2}a}$; and conse-

quently $\frac{x}{CD} \times w = \frac{mF}{\frac{1}{2}a} \times w$: But, since CF bisects the Angle mCR, we also have, $r+x$ (CR+Cm) : x

(Cm) :: $\sqrt{r^2-x^2}$ (Rm) : Fm = $\frac{x\sqrt{r^2-x^2}}{r+x} =$

$x\sqrt{\frac{r-x}{r+x}}$: Whence the Force $\frac{mF}{\frac{1}{2}a} \times w$, acting upon the Supporter, is likewise truly expressed by

$\frac{wx}{\frac{1}{2}a} \sqrt{\frac{r-x}{r+x}}$: Whereof the Fluxion being taken and

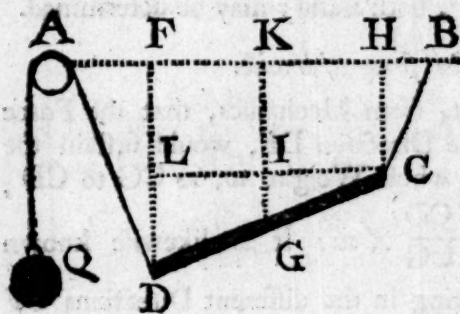
put equal to Nothing &c. we get $x = \frac{r\sqrt{5-r}}{2}$:

Therefore CR : Cm (:: $1 : \frac{\sqrt{5-1}}{2}$) :: Radius : Co-
fine of RCG = $51^\circ : 50'$, the Inclination required.

PROB.

P R O B. XXI.

451. To determine the Position of a Beam CD, moveable about one End C as a Center, and sustained at the other End D by a given Weight Q, appended to a Cord QAD passing over a Pulley at a given Point A.



Let G be the Center of Gravity of the Beam; also let DF, GK and CH be perpendicular to the Plane of the Horizon, and CL and AH parallel to the same: Put

ting $AH=a$, $CH=b$, $CD=c$, $CG=d$, $DL=x$, $CL=y$, and the Weight of the Beam $=w$. Then $AF=a-y$, $DF=b+x$, and $AD(\sqrt{AF^2+DF^2})=\sqrt{a^2-2ay+y^2+b^2+2bx+x^2}$; which (because $y^2+x^2=c^2$) will also be $=\sqrt{a^2+b^2+c^2+2bx-2ay}=\sqrt{f^2+2bx-2ay}$ (by putting $f^2=a^2+b^2+c^2$) whose

Fluxion, $\frac{bx-ay}{\sqrt{f^2+2bx-2ay}}$, multiply'd by $\mathcal{Q}$, is the

Momentum of the Weight $\mathcal{Q}$, supposing the Beam to be in Motion. Moreover, because $DC:DL::CG:$

GI , we have $GI=\frac{dx}{c}$; whose Fluxion, $\frac{d\dot{x}}{c}$, multi-

ply'd by w , is the Momentum of the Beam itself in a vertical Direction.

Wherefore making these *Momenta* equal to each other (according to the Principles of Mechanics) we get

$$\frac{bx-ay}{\sqrt{f^2+2bx-2ay}} \times \mathcal{Q} = \frac{d\dot{x}}{c} \times w, \text{ and consequently}$$

$$bx-ay \times c\mathcal{Q} = dw\dot{x} \sqrt{f^2+2bx-2ay} : \text{ But, since } y^2 +$$

$y^2 + x^2 = c^2$, we have $2yy + 2x\dot{x} = 0$, or $-\dot{y} = \frac{x\dot{x}}{y}$: And therefore (by Substitution) $b\dot{x} + \frac{ax\dot{x}}{y} \times c\mathcal{Q} = dw\dot{x} \sqrt{f^2 + 2bx - 2ay}$, or $\overline{by + ax} \times c\mathcal{Q} = dyw \times \sqrt{f^2 + 2bx - 2ay}$: From whence, and the foregoing Equation $x^2 + y^2 = c^2$, both x and y may be determined.

The same otherwise.

452. It is evident, from Mechanics, that the Force which, acting in the Direction DF, would sustain the End D, is to the whole Weight w , as CG to CD;

and therefore is $= \frac{CD}{CG} \times w$: It is likewise known

that two Forces acting in the different Directions DF and DA, so as to have the same Effect in sustaining DC, or causing It to move about the Point C, must be to each other, inversely, as the Sines of the Angles of Incidence FDC and ADC. Therefore we have $S. FDC$

: $S. ADC :: \mathcal{Q} : \frac{CD}{CG} \times w$; from which given Ratio of the Sines, the Angles themselves will be found, by an algebraic Process independent of Fluxions.

COROLLARY.

453. If the Position of CD be supposed given, and the Tension of AD (or the Weight $\mathcal{Q}$) be required: Then, from the foregoing Proportion, we shall have $\mathcal{Q} =$

$\frac{S. FDC}{S. ADC} \times \frac{CG}{CD} \times w$. Which will also express the

Tension of AD when the End C is sustained by a Cord BC instead of a Pin at C: Whence it follows that the Tensions of two Cords AD and BC, sustaining a Beam or Rod CD, at its Extremes D and C, are expressed by

$\frac{S. FDC}{S. ADC} \times \frac{CG}{CD} \times w$, and $\frac{S. HCD}{S. BCD} \times \frac{DG}{CD} \times w$; and

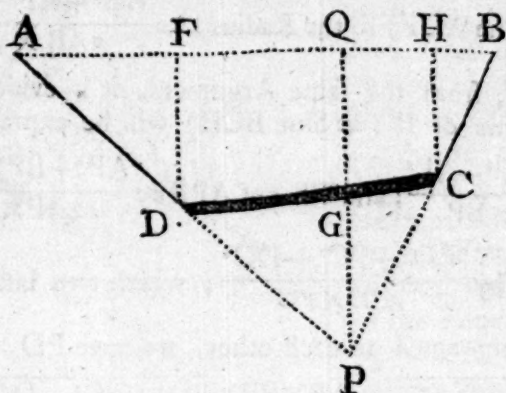
there-

therefore are to each other as $\frac{CG}{S. ADC}$ to $\frac{DG}{S. BCD}$ or
as $S. BCD \times CG$ to $S. ADC \times DG$ respectively; be-
cause the Sine of FDC and that of its Supplement HCD
are equal to each other.

P R O B. XXII.

454. To determine the Position of a Beam DC, sus-
pended at its Extremes by two Cords AD and BC of given
Lengths, from two given Points A and B in the same
horizontal Line AB.

Let G be the Center of Gravity of the Beam, and
let DF and CH be perpendicular to AB.



It appears, from the Corol. to the last Problem, that
the Tension of AD is to that of BC, as $\frac{CG}{S. ADC}$ to

$\frac{DG}{S. BCD}$; whence (by the Resolution of Forces) the

Force of AD, in a Direction parallel to the Horizon,
is to the Force of BC, in the opposite Direction, as

$\frac{CG}{S. ADC} \times \frac{S. ADF}{Rad.}$ to $\frac{DG}{S. BCD} \times \frac{S. BCH}{Rad.}$. Which

Forces, that the Beam may remain in *Equilibrio*, must

M m

con-

consequently be equal to each other ; and therefore

$$\frac{S. BCD}{S. ADC} = \frac{S. BCH}{S. ADF} \times \frac{DG}{CG}. \text{ But now, to determine}$$

the Angles themselves, from this Equation and the given Lengths of AB, BC &c. let AD and BC be produced to meet each other in P, and let PQ, perpendicular to AB, be drawn ; putting $AB = a$, $AD = b$, $BC = c$, $DC = d$, $DG = f$, $CG = g$, $AP = x$ and $BP = y$.

Then, because $AB : AP + BP :: AP - BP : AQ - BQ$

$$= \frac{AP^2 - BP^2}{AB}, \text{ we have } AQ = \frac{1}{2} AB + \frac{AP^2 - BP^2}{2AB}$$

$$= \frac{AB^2 + AP^2 - BP^2}{2AB}; \text{ and consequently the Co-sine of}$$

$$A (= \text{Sine ADF}) \text{ to the Radius } 1 = \frac{AB^2 + AP^2 - BP^2}{2AB \times AP}:$$

Whence, from the same Argument, it is evident that the Co-sine of B (= Sine BCH) will be expressed by

$$\frac{AB^2 + BP^2 - AP^2}{2AB \times BP}; \text{ and That of APB by } \frac{AP^2 + BP^2 - AB^2}{2AP \times BP};$$

$$\text{and also by } \frac{PD^2 + PC^2 - DC^2}{2PD \times PC}; \text{ which two last Quan-}$$

tities being equal to each other, we have $PD \times PC \times \frac{AP^2 + BP^2 - AB^2}{2AB} = AP \times BP \times \frac{PD^2 + PC^2 - DC^2}{2PD \times PC}$; that

is $x - b \times y - c \times x^2 + y^2 - a^2 = xy \times x - b^2 + y - c^2 - d^2$.

Moreover, since $PC : PD :: S. ADC \text{ (or PDC)} : S. BCD$

$$\text{(or PCD) we also have } \frac{PD}{PC} = \frac{S. BCD}{S. ADC} = \frac{S. BCH}{S. ADF} \times$$

$$\frac{DG}{CG} \text{ (by the first Equation); whence } CG \times PD \times$$

$S. ADF = DG \times PC \times S. BCH$; that is $CG \times PD \times$

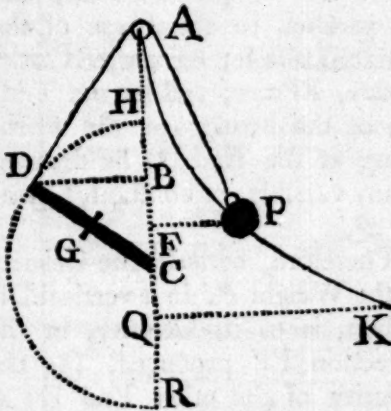
$$\frac{AB^2 + AP^2 - BP^2}{2AB \times AP} = DG \times PC \times \frac{AB^2 + BP^2 - AP^2}{2AB \times BP}, \text{ or}$$

CG

$CG \times PD \times BP \times \overline{AB^2 + AP^2 - BP^2} = DG \times PC \times AP \times \overline{AB^2 + BP^2 - AP^2}$, which, in algebraic Terms, is $gy \times x - b \times a^2 + x^2 - y^2 = fx \times y - c \times a^2 + y^2 - x^2$. From whence and the preceding Equation the Values of x and y will be known.

PROB. XXIII.

455. *Supposing a Beam CD, moveable about one End C, as a Center, to be sustained at the other End D by means of a given Weight P, hanging at a Rope passing over a Pulley at a given Point A, vertical to C; 'tis proposed to find the Curve APK along which the Weight must ascend, or descend, so as to be, every where, a just Counterpoise to the Beam.*



From the Center C, with the Radius CD, let a Semi-circle HDR be described, and let DB and PF be perpendicular to the vertical Line AHCR; also let $CD = a$, $CA = b$, $AH = c$, $AF = x$, $PF = y$, $HB = z$, and the Length of the Rope $DAP = m$; likewise let $HQ (b)$ be the given Value of x

(AF) when D coincides with H.

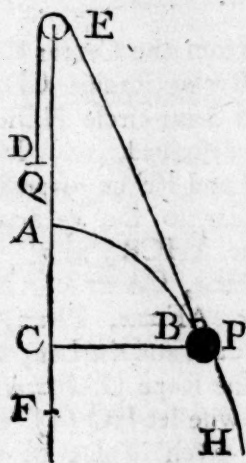
Because the Weight and the Beam are always in *Equilibrio*, by Hypothesis, their *Momenta*, and consequently their Velocities, in a vertical Direction, must be every where in a constant Ratio; and therefore the Distance QF ($b - x$) ascended by the Weight P, will be, to the Distance HB descended by the End of the Beam D likewise in a constant Ratio: Let this Ratio be that of b to any given Quantity d , that is, let $b - x : z :: b : d$, and we shall have $db - dx = bz$: Moreover, we have $AD^2 (CD^2 + AC^2 - 2AC \times BC) = a^2 + b^2 - 2b \times a - z = b - a^2 + 2bz = c^2 + 2bz = c^2 - 2db + 2dx$: Whence $AP (m - AD) = m -$

$\sqrt{cc-2db+2dx}$, and therefore, $y^2 (AP^2 - AF^2) = m - \sqrt{cc-2db+2dx}^2 - x^2$. Q. E. I.

After the same manner a Curve may be found, along which a Weight descending, shall be every where in *Equilibrio* with another Weight ascending thro' the Arch of a given Curve.

PROB. XXIV.

456. To find the Equation of a Curve ABH, along which a given Weight P, suspended by a String PED passing over a Pulley E, must descend, so that the Tension of the String may vary according to any given Law.



Let EC be perpendicular, and CP parallel, to the Plane of the Horizon; also let $AE=a$, $AC=x$, $CB=y$, $EP=v$, and let the Tension of the String (or the Force acting at the End D) be denoted by any variable, or constant, Quantity $\mathcal{Q}$.

Therefore, because the Celerity of the Weight P, in a vertical Direction, is to its Celerity, in the Direction EP produced, (or the Celerity of the other End D) as $\dot{x}$ to $\dot{v}$, it is evident that the Weight itself must be to the tending Force $\mathcal{Q}$, inversely in that Ratio, and consequently $P\dot{x} = \mathcal{Q}\dot{v}$.

Furthermore, because $EC=a+x$ and $BC^2 = BE^2 - EC^2$, we have $y^2 = v^2 - a^2 - x^2$: From which Equations, when the Relation of P and $\mathcal{Q}$ is given, the Curve itself will also be known.

Thus, for Example, let the Ratio of P to $\mathcal{Q}$, be constant, or that of m to n, then $m\dot{x}$ being $=n\dot{v}$, we have (by taking the Fluent) $mx + na = nv$; whence

$$v = a + \frac{mx}{n}; \text{ and therefore } y^2 (=a^2 + \frac{2max}{n} + \frac{m^2x^2}{n^2})$$

$$-a^2 - 2ax - x^2) = \frac{m-n}{n} \times 2ax + \frac{m^2-n^2}{n^2} \times x^2:$$

Which is the Equation of an Hyperbola.

Again, for a second Example, let the tending Force $\mathcal{Q}$ be to the Weight P , as DE^n to $AC^m \times c^{n-m}$, or as $\overline{b-v}^n : x^m c^{n-m}$ (supposing $b = PED$ and $c =$ any given Line AF) Therefore, since $\mathcal{Q} = \frac{\overline{b-v}^n}{c^{n-m} x^m} \times P$, and

$$\frac{\overline{b-v}^n}{c^{n-m} x^m} \times P \dot{v} (= \mathcal{Q} \dot{v}) = P \dot{x}, \text{ we have } \overline{b-v}^n \times \dot{v}$$

$$= c^{n-m} x^m \dot{x}, \text{ and so } \frac{\overline{b-v}^{n+1} - \overline{b-v}^{n+1}}{n+1} =$$

$$\frac{c^{n-m} x^{m+1}}{m+1}; \text{ whence } \overline{b-v}^{n+1} = \overline{b-a}^{n+1} -$$

$$\frac{n+1 \times c^{n-m} x^{m+1}}{m+1}, \text{ and } v \text{ (EP)} = b -$$

$$\left(\frac{\overline{b-a}^{n+1} - \frac{n+1 \times c^{n-m} x^{m+1}}{m+1}}{n+1} \right)^{\frac{1}{n+1}}. \text{ From which}$$

the Relation of x and y , or the Value of BC , is also known.

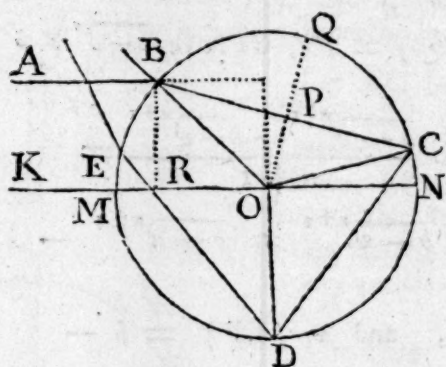
But if $m = 0$, and $n = 1$, (which will be the Case when the Force acting at D is equal to that by which a Beam or Rod is made to move about a Center, as in the last Problem) v will then become, barely, $= b -$

$$\overline{b-a}^2 - 2cx)^{\frac{1}{2}}, \text{ and therefore } y^2 (= v^2 - a^2 + x^2)$$

$$= b - \sqrt{\overline{b-a}^2 - 2cx} - a + x^2: \text{ Therefore } ABH \text{ is, in this Case, a Line of the fourth Order.}$$

PROB. XXV.

457. *Supposing a Ray of Light ABCD to be refracted at the Surface of a given Sphere MQND, and afterwards reflected any given Number (n) of Times, within the Sphere; to determine the Distance of the Incident Ray AB from the Axis MN, so that the Arch MBCDE, intercepted by the given Point M and the emerging Ray at E, may be a Minimum.*



Let the Radius $OB = 1$, the Sine of Incidence $BR = x$, and the Sine of Refraction $OP = y$, and let the given Ratio of the two last be that of p to q .

Since all the Angles of Incidence and Reflexion BCO , OCD , CDO &c. are equal, the Arcs

BC , CD and DE must also be equal; and consequently

$$MBCDE = MB + \overline{n+1} \times BC = MB + \overline{2n+2} \times BQ:$$

• Art. 22. Whose Fluxion is to be equal to Nothing*. Now the Fluxion of the Arch MB , whose Sine is x and

† Art. 142. Radius Unity, will be $= \frac{\dot{x}}{\sqrt{1-x^2}} \dagger$; and that of

the Arch BQ , whose Co sine (OP) is y , $= \frac{-\dot{y}}{\sqrt{1-y^2}}$,

Hence we have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times \dot{y}}{\sqrt{1-y^2}} = 0$: But,

since $x:y::p:q$, y is $= \frac{qx}{p}$ and $\dot{y} = \frac{q\dot{x}}{p}$; and so we

have $\frac{\dot{x}}{\sqrt{1-x^2}} - \frac{\overline{2n+2} \times q\dot{x}}{\sqrt{p^2 - q^2x^2}} = 0$; whence (putting

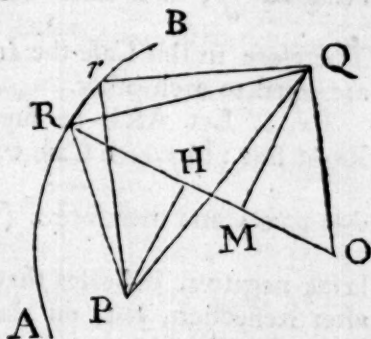
$m =$

$m = 2n+2$ x is found $= \frac{1}{q} \sqrt{\frac{m^2 q^2 - p^2}{m^2 - 1}}$: From which it is observable, that, when mq is less than p , or $2n+2$ less than $\frac{p}{q}$, the Arch MBCD continually increases with BM; and therefore is the least possible, when B coincides with M. Q. E. I.

P R O B. XXVI.

458. If two Rays of Light PR and Pr, from a given Point P, making an indefinitely small Angle with each other, be reflected at a given Curve Surface ARB; 'tis proposed to determine the Concourse, or Focus, Q of the reflected Rays RQ and rQ.

Let RO, perpendicular to the Curve, be the Radius of a Circle having the same Curvature with ARB at R; make PH and QM perpendicular to RO, join Q, O; and put RO $= r$, PR $= y$, RH $= v$, and RQ $= z$.



Then, because the Angle of Reflection ORQ is equal to the Angle of Incidence ORP, the Triangles RQM and RPH will be similar, and therefore $y : v :: z : RM$

$$= \frac{vz}{y} : \text{Whence } OQ^2 (RO^2 + RQ^2 - 2RO \times RM)$$

$$= r^2 + z^2 - \frac{2rvz}{y}.$$

But, since this Quantity OQ^2 continues the same (by Hypothesis) whether we regard one Ray or the other (that is, whether y stands for PR or Pr) its Fluxion must therefore be equal to Nothing; that

M m 4

is,

is $2xz = \frac{2rvzy + 2rvzy - 2rvzy}{y^2} = 0$: Whence

$$z = \frac{vyz}{\frac{y^2z}{r} + vy - yv} : \text{But (by Art. 35.) } z = -y; \text{ therefore } z =$$

$$- \frac{\frac{-vy}{y^2j}}{\frac{y^2j}{r} + vy - yv} : \text{Moreover (by Art. 73.) } r = \frac{yy}{v};$$

therefore

$$z = \frac{-vy}{-yv + vy - yv} = \frac{vy}{2yv - vy} . \quad \text{Q. E. I.}$$

Example 1. Let ARB be an Arch of the Logarithmic Spiral: Whose Equation is $av = by \dagger$: And then, v being $= \frac{by}{a}$, we shall have $z \left(\frac{vy}{2yv - vy} \right) = y$:

Therefore in this Case the Incident and Reflected Rays are equal to each other.

Ex. 2. Let ARB be supposed to degenerate into a Right-line: In which Case v being constant, its Fluxion $\dot{v}$ is $= 0$; and therefore $z \left(= \frac{vy}{-vj} \right) = -y$: Which

being negative, indicates that the Rays do not converge after Reflection, but, on the contrary, diverge from a Point on the contrary Side of ARB, at the Distance y . Which is very easy to demonstrate by common Geometry.

P R O B. XXVII.

459. Let two Rays of Light PR and Pr, from a given Point P, be refracted at a given Curve Surface ARB; to determine the Focus Q of the refracted Rays RQ and rQ.

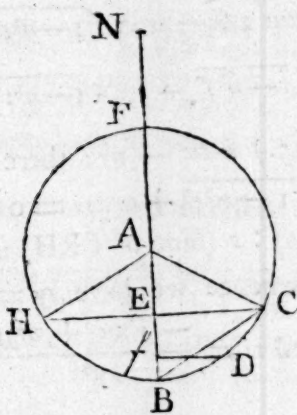
Let the Lines RO, RH &c. be drawn, and denoted as in the preceding Problem: Moreover, let the Sine of Incidence PRH (to the Radius 1) be represented by s , and let it be to the Sine of Refraction ORQ, in the given Ratio of 1 to n .

$nzv^2\dot{y} - nyzv\dot{v} = 0$. But (by Art. 73.) $r = \frac{y\dot{y}}{v\dot{v}}$,
 therefore $-zyw\dot{v} + w^2y\dot{y} + nzv^2\dot{y} - nyzv\dot{v} = 0$, and
 consequently $z = \frac{w^2y\dot{y}}{wy + nv\dot{y} \times \dot{v} - nv^2\dot{y}}$. Q. E. I.

From this Solution, that of the preceding Problem is easily derived: Also from hence the Caustic (or the Curve which is the Locus of all the Points Q thus found) will likewise be given.

P R O B. XXVIII.

460. To find the Time of the Vibration of a Pendulum in the Arch of a Circle.



• Art. 142.

Let AB denote the Pendulum in a vertical Position; and from any Point D in the given Arch CBH, wherein the Vibrations are perform'd, draw Df parallel to CH; and let $AB=a$, $BE=c$, $Bf=x$, and $BD=z$: By the Nature of the Circle we have $z =$

$$\frac{a\dot{x}}{\sqrt{2ax-xx}}^* : \text{Whence the}$$

Fluxion of the Time, being

† Art. 207. as $\frac{z}{\sqrt{Ef}}$ †, will be defined by $\frac{a\dot{x}}{\sqrt{c-x} \times \sqrt{2ax-xx}}$

$$= \frac{a\dot{x}}{\sqrt{cx-xx} \times \sqrt{2a-x}} = \frac{\frac{1}{2}a^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx-xx} \times \sqrt{1-\frac{x}{2a}}}$$

$$= \frac{\frac{1}{2}a^{\frac{1}{2}} \times \dot{x}}{\sqrt{cx-xx}} \times \left(1 + \frac{x}{2 \cdot 2a} + \frac{3x^2}{2 \cdot 4 \cdot 4a^2} + \frac{3 \cdot 5x^3}{2 \cdot 4 \cdot 6 \cdot 8a^3} \right)$$

$$+ \frac{3 \cdot 5 \cdot 7x^4}{2 \cdot 4 \cdot 6 \cdot 8 \cdot 16a^4} \text{ \&c. Whereof the Fluent,}$$

when

when $x=c$, (or $\sqrt{cx-x^2}^{\frac{1}{2}}=0$) is, (by *Art.* 142. and 286.)

$$\text{equal to } p\sqrt{\frac{1}{2}a} \times 1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 2a^2} \\ + \frac{3 \cdot 3 \cdot 5 \cdot 5c^3}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 2a^3} + \frac{3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7c^4}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 2a^4}$$

&c. Which therefore is proportional to the Time of half one Vibration; where p stands for the Semi-Periphery of the Circle whose Radius is Unity.

COROLLARY I.

461. Since the Time of the perpendicular Descent of a Body through any given Right-line u , computed according to the same Method, is as the Fluent of

$$\frac{u}{\sqrt{u}} \text{ or } 2\sqrt{u}, \text{ it follows that the Time of falling}$$

along the Diameter BF ($2a$), or the Chord CB †, will † *Art.* 205.

be truly defined by $2\sqrt{2a}$: Which therefore is to the

Time of the Descent thro' the Arch CDB, as $\frac{4}{p}$ to 1

$$+ \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 2a^2} \text{ \&c. From whence,}$$

as the Time of falling thro' the Diameter BF, is absolutely given, by *Art.* 202. the true Time of Vibration will also be known.

COROLLARY II.

462. If the Arch in which the Pendulum vibrates be very small, the above Proportion will become, nearly, as 4 to p : From which it appears, that the Time of Descent thro' any very small Arch CB is to that along the Chord CB, as the Periphery of any Circle is to four times its Diameter.

COROLLARY III.

463. Hence, we have a Method for determining how far a Body freely descends in a given Time; by knowing the

the Time of Vibration, of a given Pendulum: For, if BN be assumed for the Space thro' which a Body would descend during the Time of one whole Vibration, in the very small Arch CBH; then, the Distances descended being as the Squares of the * Times we have, from the last Corollary, as $4^2 : 2p^2 :: BF (2a) : BN$; or $1 : \frac{1}{2}p^2 : a : BN$; that is, as the Square of the Diameter of a Circle is to half the Square of its Periphery, so is the Length of the Pendulum, to the Distance a Body will freely descend, from Rest, in the Time of one Oscillation. Thus, for instance (because it is found from Experiment that a Pendulum 39,2 Inches long vibrates Seconds) it will be as $1 : 4,934 (= \frac{1}{2}p^2) :: 39,2 : 193$ Inches, the Distance which a heavy Body will fall in the first Second of Time.

COROLLARY IV.

464. Moreover, from the foregoing Series, the Time which a Pendulum, vibrating in an exceeding small Arch, will lose when made to vibrate in a greater Arch of the same Circle may also be deduced:

For let T be put to denote the Number of Seconds in 24 Hours (or any other given Time) then the Number of Vibrations, performed in that Time will be as

$$\frac{T}{1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot 2a^2} \&c.}; \text{ which, there-}$$

fore, in an exceeding small Arch (where c may be taken as Nothing) will be expressed by T : And so the Time (t) or Number of Vibrations lost will be $T -$

$$\frac{T}{1 + \frac{c}{2 \cdot 2 \cdot 2a} + \frac{3 \cdot 3c^2}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 2a^2} \&c.} = T \times$$

$$\frac{c}{8a} + \frac{5c^2}{256a^2} \&c. \text{ (by dividing by the Denominator.)}$$

Now, if the Number of Degrees described on each Side of the Perpendicular be represented by D , the Arch

Arch itself, on each Side, will be $= 3.14159 \text{ } \&^c. \times a \times \frac{D}{180}$; which, if the Value of D be not more than about 15 or 20 Degrees, will be nearly equal to its Chord, represented by $\sqrt{2ac}$ ($= \sqrt{BF \times BE}$.) From which Equation we get $\frac{c}{a} = \frac{D^2}{6560}$: This Value, sub-

stituted above, gives $t = T \times \frac{D^2}{8 \times 6560} + \frac{5D^4}{256 \times 6560^2} \text{ } \&^c.$
 $= T \times \frac{D^2}{52480}$ nearly: Which, when T is interpreted

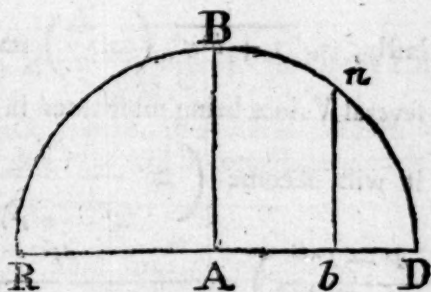
by 86400 Seconds (or one whole Day) becomes $= 1\frac{1}{2} \times D^2$, nearly: And so many are the Seconds which will be lost *per Diem* in the Arch D . From whence we gather, that if the Pendulum measures true Time in any small Arch, whose Degrees on each Side the Perpendicular are denoted by A , the Number of Seconds lost *per Diem* in another Arch whose Degrees are B , will be nearly

represented by $\frac{3}{2} \times \overline{B^2 - A^2}$: Thus, if a Pendulum measures true Time, in an Arch of 3 Degrees, it will lose $10\frac{1}{2}$ Seconds a Day in an Arch of 4 Degrees, and $24''$ in an Arch of 5 Degrees.

P R O B. XXIX.

465. To determine the Meridional Parts answering to any proposed Latitude, according to Wright's Projection, apply'd to the true spheroidal Figure of the Earth.

Let DAR be the Axis, AB the Semi-equatoreal Diameter, and DBR a Meridian, of the Earth; also let bn be an Ordinate to the Ellipsis DBR; putting $AD (= AR)$



$=1$, $BA=d$, $Ab=x$, $bn=y$, $Bn=z$, and the Meridional Distance (in Parts of the Semi-Axis AD) $=u$.

Then, by the Nature of the Ellipsis, we have $y=d$

$\sqrt{1-x^2}$; therefore $\dot{y} = \frac{-dx\dot{x}}{\sqrt{1-x^2}}$; and consequently

$\dot{z} = \sqrt{\dot{x}^2 + \frac{d^2 x^2 \dot{x}^2}{1-xx}}$: Which, by putting $b^2 = d^2$

-1 , will be reduced to $\dot{z} = \frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-x^2}}$. Whence,

by the Nature of the Projection, it will be as bn

$(d\sqrt{1-x^2}) : AB (d) :: \dot{z} \left(\frac{\dot{x}\sqrt{1+b^2x^2}}{\sqrt{1-xx}} \right) : \dot{u} =$

$\frac{\dot{x}\sqrt{1+b^2x^2}}{1-x^2}$; which is the Fluxion of the Quantity

required: But we are now to get the same thing expressed in Terms of the Latitude of the Place n : In order thereto, putting the Sine of that Latitude $=s$, we

have, by Trigonometry, as $\dot{z} \left(\frac{x\sqrt{1+b^2x^2}}{\sqrt{1-x^2}} \right) : -\dot{y}$

$\left(\frac{dx\dot{x}}{\sqrt{1-x^2}} \right) :: \text{Radius } (1) : s$; and consequently

$s\sqrt{1+b^2x^2} = dx$; from which Equation x is found $=$

$\frac{s}{\sqrt{d^2 - b^2s^2}}$: Whence $\dot{x} = \frac{d^2\dot{s}}{d^2 - b^2s^2}^{\frac{3}{2}}$; also $1-x^2 =$

$\frac{d^2 - b^2s^2 - s^2}{d^2 - b^2s^2} = \frac{d^2 - d^2s^2}{d^2 - b^2s^2}$ (because $d^2 = 1 + b^2$) and,

lastly, $\sqrt{1+b^2x^2} \left(= \frac{dx}{s} \right) = \frac{d}{\sqrt{d^2 - b^2s^2}}$: Which

several Values being substituted in that of $\dot{u}$, found above,

it will become $\left(= \frac{d^2\dot{s}}{d^2 - b^2s^2}^{\frac{3}{2}} \times \frac{d}{\sqrt{d^2 - b^2s^2}} \times \right.$

$\left. \frac{d^2 - b^2s^2}{d^2 \times 1 - ss} \right) = \frac{d\dot{s}}{d^2 - b^2s^2 \times 1 - ss}$; which resolved

into two Parts, for the more readily finding the Fluent,

gives $u = \frac{ds}{1-s^2} - \frac{db^2s}{d^2-b^2s^2}$: Whereof the Fluent being taken, we have

$$u = \begin{cases} 2.302585 \text{ \&sc.} \times \frac{1}{2} d \times \text{Log.} \frac{1+s}{1-s} \\ - 2.302585 \text{ \&sc.} \times \frac{1}{2} b \times \text{Log.} \frac{d+bs}{d-bs} \end{cases}$$

But, as 3,14159 &sc. $\times 2d$ (the Measure of the whole Periphery of the Earth at the Equator, in Parts of the Semi-Axis AD) is to 21600 (the Measure of the same Periphery in Geographical Miles) so is the foresaid Value of u to

$$\left\{ \begin{array}{l} 3958 \times \text{Log.} \frac{1+s}{1-s} \\ - \frac{3958b}{d} \times \text{Log.} \frac{d+bs}{d-bs} \end{array} \right\} \text{ the corresponding Value}$$

of u , in Geographical Miles, or the Meridional Parts required.

COROLLARY.

466. If the Earth be considered as differing but little from a Sphere, d will be nearly $=1$, and consequently $(\sqrt{d^2-1})$ the Value of b , very small: Therefore, in this Case, the latter Part of our Fluent $\left(-\frac{3958b}{d} \times \text{Log.} \frac{d+bs}{d-bs}\right)$ will become nearly $= 3440b^2s$ (because $\text{Log.} \frac{d+bs}{d-bs} = \frac{2bs}{d}$) $\times \frac{1}{2.3025 \text{ \&sc.}}$ †. But if the Earth be taken as a perfect Sphere, this last Expression will vanish, and so the Value of u will become barely $= 3958$

† There is a Mistake in p. 43. and 44. of my Dissertations (by forgetting to divide by the Modulus 2.3025 &c.) which may from hence be rectify'd.

$\times \text{Log. } \frac{1+s}{1-s}$. Which Logarithm, it is easy to prove,

expresses twice the artificial Tangent of half the given Latitude increased by 45 Degrees (Radius being Unity.) Wherefore, if the Meridional Parts answering to any given Latitude, thus found (from a Table of logarithmic Tangents) when the Earth is considered as a perfect Sphere, be denoted by M , it follows that the Meridional Parts answering to the same Latitude, when the Earth is taken as a Spheroid, will be nearly equal to $M - 3440b^2$: Which, because $OP^2 (1) : OA^2 (1+b^2) ::$
 * Art. 397. $230 : 231$ *, will (by substituting the Value of b hence arising) be reduced to $M - 30s$. Whence the following Rule.

M

As Radius, to the Sine of the given Latitude, so is 30 to a Fourth-Proportional; which subtracted from the Meridional Parts when the Earth is taken as a Sphere (found as above) gives the Meridional Parts answering to the same Latitude, when it is considered as an oblate Spheroid.

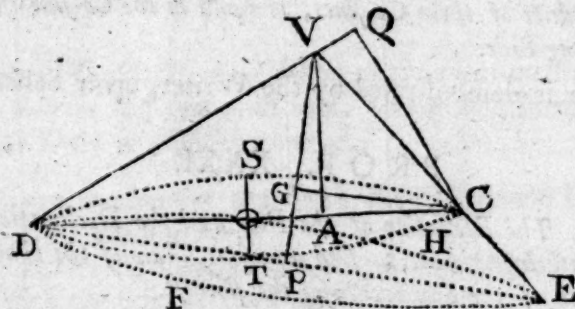
Thus, for Example, let the given Latitude be 50° : Then, first, for the Meridional Parts in the Sphere; we must, according to the foregoing Prescript, take the Logarithmic Tangent of $25^\circ + 45^\circ$, or 70° : Which, by the Table, is found $= 0,43893$ &c. This multiply'd by the constant Multiplier 7916 ($= 2 \times 3958$) produces 3475 for the Meridional Parts in the Sphere: Then by the Rule above, it will be as Radius to the Sine of 50° , so is 30 to 23; which subtracted from 3475, leaves 3452 for the Meridional Parts answering to 50° Latitude, in the Spheroid.

P R O B. XXX.

467. *To determine the Paths which Shadows of Objects describe, upon the Plane of the Horizon, during the Sun's apparent diurnal Revolution.*

Let CSODT be the Plane of the Horizon, and AV the perpendicular Height of the Object: Then, since the Rays, intercepted by the highest Point V, would, in the Sun's diurnal Revolution, form a conical Surface

face VDFEH about that Point as a Vertex ; whose Axis PV produced passes thro' the Pole of the World ; it is evident that the Path of the Shadow, being the Intersection of the Plane of the Horizon with that Surface, must be a Conic Section.



Let its two principal Diameters therefore (when an Ellipsis, that is, when the Sun never descends below the Horizon) be CD and ST; also let DPE and CG be perpendicular to VP the Axis of the Cone, and CQ perpendicular to DV: Putting the Sine of (QVC) twice the Sun's Declination $VEP=f$; the Sine of (DCV) his greater Meridional Altitude $=g$, and that of the lesser (CDV) $=b$: Then (by plane Trig.) $g:1 \text{ (AV)} :: 1$

{Radius} : CV = $\frac{1}{g}$; and h (Sine of CDV) : $\frac{1}{g}$ (CV)

$$\therefore f \text{ (Sine of DVC)} : DC = \frac{f}{g_b} : \text{Moreover, } \mathbf{r} \text{ (Ra-}$$
$$\text{dius} : \frac{r}{g} \text{ (CV)} :: p \text{ (the Sine of the Comp. Decl.)}$$

GVC) : GC = $\frac{p}{g}$: And in the very same Manner it

will be found that $DP = \frac{p}{h}$: But $GC \times DP = OS^2$

(*vid. Art. 41.*) whence we have $ST(2OS) = \frac{2p}{\sqrt{gb}}$:

From which, and the Transverse Axis ($DC = \frac{f}{g^2h}$) the

Curve itself is given.

Q. E. I.

LEMMA

Na

LEMMA.

468. In any spherical Triangle, if Radius be supposed Unity, the Product of the Sines of any two of the Sides drawn into the Co-sine of the Angle they include, added to the Product of their Co-sines, is equal to the Co-sine of the remaining Side.

This is demonstrated by the Writers upon Spherics.

PROB. XXXI.

469. The Elevation of the Pole and the Declination of the Sun being given, to find at what Time of the Day the Azimuth of the Sun increases the slowest.



It is evident that the Time sought will be when the Fluxion of the Hour Angle P, bears the greatest Ratio possible to That of the Azimuth Z.

Now the Fluxion of the Angle P is to that of Z, universally, as $\text{Rad.} \times S.ZO : S.PO \times \text{Co-f. } O$ (by Art. 256. Case 2.) Consequently

$\frac{S.PO \times \text{Co-f. } O}{\text{Rad.} \times S.ZO}$, or $\frac{\text{Co-f. } O}{S.ZO}$ is a Minimum, in this

Case, because PO may be considered as constant.

Let now the Sine of PO be put $=p$, its Co-sine $=d$, the Co-sine of PZ $=b$, that of ZO $=x$, and that of O $=y$; then, the Sine of ZO being $=\sqrt{1-x^2}$, we have (by the Lemma) $p\sqrt{1-x^2} \times y + dx = b$; whence

$$y = \frac{b-dx}{p\sqrt{1-x^2}}, \text{ and therefore } \frac{\text{Co-f. } O}{S.ZO} \left(= \frac{y}{\sqrt{1-x^2}} \right) \\ = \frac{b-dx}{p \times 1-x^2} : \text{ Which put into Fluxions, and re-}$$

duced,

duced, gives $x = \frac{b - \sqrt{b^2 - d^2}}{d}$, for the Sine of the

Sun's Altitude at the Time required : Whence the Time itself is given.

PROB. XXXII.

470. To determine the Ratio of the Heat received from the Sun in different Latitudes, during the Time of one whole Day, or any Part thereof.

Let p = the Sine of the Sun's Polar-Distance $P\odot$ (see the last Fig.)

d = its Co-sine, or the Sine of the Declination.

b = the Sine of the Pole's Elevation.

c = its Co-sine, or the Sine of PZ .

z = the Angle (P) expressing the Time from Noon.

x = its Sine, and $\sqrt{1-x^2}$ = its Co-sine.

Then (by the foregoing Lemma) we shall have $pc\sqrt{1-x^2} + bd = \text{Co-sine } Z\odot = \text{Sine of the Sun's Altitude}$

Now, it is known that the Number of Rays falling in any given Particle of Time, upon a given horizontal Plane, is as that Time and the Sine of the Sun's Altitude conjunctly : Therefore the Number of Rays falling

in the Time z , or $\frac{z}{\sqrt{1-xx}}$ (vid. Art. 142.) will

be defined by $pcx + bdz$: Whose Fluent $pcx + bdz$ is, therefore, as the Heat required.

Where it may be observed,

1. That when the Latitude and Declination are of different Kinds, or $P\odot$ is greater than 90 Degrees, the Value of d is to be considered as a negative Quantity.

2. That, if the Expression for the Heat found above be divided by the Square of the Sun's Distance from the Earth, the Quotient will exhibit the Ratio of the Heat, allowing for the Excentricity of the Earth's Orbit.

COROLLARY I.

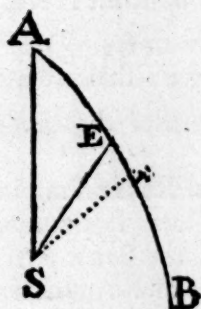
471. If the Place proposed be at the Equator, the Heat, received in half one diurnal Revolution, will be barely as p ; because $b=0$, $c=1$, and $x=1$.

COROLLARY II.

472. But if the Place be at the Pole, then the Heat will be as $d \times \overline{3,14159} \text{ \&c.}$ since, in this Case, $c=0$, $b=1$, and z (= Semi-Circle) $= 3,14159 \text{ \&c.}$

LEMMA.

473. The Number of Particles of Light, ejected by the Sun, upon the Earth, in a given Time, is proportional to the Angle described about his Center in that Time.



For, let S represent the Center of the Sun, AEB the Orbit of the Earth (or That of any other Planet) and let E and r be two Points therein as near as possible to each other: Since the Triangle ESr may be taken as rectilinear, its Area, if the Angle ESr be supposed given, or every where the same, will be as $SE \times Sr$, or SE^2 : And therefore the Time of describing Er

(being always as that Area) is also explicable by SE^2 : But the Intensity of the Light, or Heat, at the Distance of

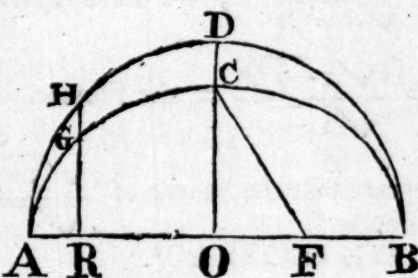
SE is as $\frac{1}{SE^2}$: Therefore the Intensity compounded with the Time (or the whole Number of Particles received in that Time) will consequently be as $\frac{1}{SE^2} \times SE^2$ (=1) Which being every where the same, the Proposition is manifest.

P R O B. XXXIII.

474. To determine the Ratio of the Heat received from the Sun at the Equator and either of the Poles, during the Time of one whole Year, or any Part thereof.

If

If the Sine of the Sun's Declination be denoted by d and its Co-sine by p , the Heat received at the Equator, and the Pole, during half one diurnal Revolution of the Sun, will be as p and $d \times 3,14159 \text{ \&c.}$ respectively (by the



Corollaries to the preceding Problem.)

Let the Sun's Longitude, considered as variable, be now denoted by z , and its Sine by s ; and let f be put for the Sine of the Obliquity of the Ecliptic: Then (per Spherics) we shall have $d=fs$, and consequently p ($=\sqrt{1-d^2}$) $=\sqrt{1-f^2s^2}$: Wherefore, seeing the Ratio of Heat in the two Places, for one Half-Day, is that of $\sqrt{1-f^2s^2}$ to $fs \times 3,14 \text{ \&c.}$ let each of these

Terms be multiply'd by $\frac{s}{\sqrt{1-s^2}}$ ($=z$) * expressing * Art. 142.

the Quantity of Heat falling upon the Earth in the Time of describing z (see the foregoing Lemma) then

the Products $\frac{s\sqrt{1-f^2s^2}}{\sqrt{1-s^2}}$, and $3,14f \times \frac{ss}{\sqrt{1-s^2}}$ will

be the Fluxions of the required Heat, answering to z .

But now to exhibit the Fluents hereof, let ACB be an Ellipsis whose greater Semi-Axis AO is = Unity, and its Excentricity FO= f ; and, supposing ADB to be a Circle described about the Ellipsis, let the Arch DH express the Sun's Longitude from the Equinoctial Point; whose Sine (OR) being = s , its Co-sine RH will be =

$$\sqrt{1-ss}.$$

But, by the Property of the Ellipsis, OD (1) OC : ($\sqrt{1-f^2}$) :: RH ($\sqrt{1-ss}$) : RG = $\sqrt{1-ff} \times \sqrt{1-ss}$: Whose Fluxion being =

Na 3

$$\frac{\sqrt{1-ff \times -ss}}{\sqrt{1-ss}}, \text{ we have } \sqrt{s^2 + \frac{1-ff \times s^2 s^2}{1-ss}}$$

$$= \frac{s \sqrt{1-f^2 s^2}}{\sqrt{1-ss}} = \text{the Fluxion of CG. Whence it}$$

appears that the Fluent of $\frac{s \sqrt{1-f^2 s^2}}{\sqrt{1-ss}}$ is truly defined by CG, or $CG \times AO^2$.

But the Fluent of the other given Fluxion, $3.14f \times \frac{ss}{\sqrt{1-ss}}$, will be $= 3.14f \times 1 - \sqrt{1-ss} = ADB \times$

$FO \times OD - RH$. Therefore the two Fluents, when H and G coincide with A, will be to each other as $CA \times AO$ to $ADB \times FO$: Whereof the Antecedent, multiply'd by 4, will be as the Heat received at the Equator during one whole Year; and the Consequent, multiply'd by 2, as the Heat at the Pole in the same Time (because the Sun shines at the Pole only two Quarters of the Year.) Hence the required Ratio, of the Heat received at the Equator and Pole, in one whole Year, will be That of $CA \times AO$ to $DAXFO$;

or, in Species, as $1 - \frac{f^2}{2.2} - \frac{3f^4}{2.2.4.4} - \frac{3.3.5f^6}{2.2.4.4.6.6}$

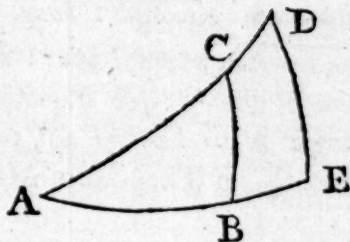
• Art. 434* &c. to f ; which, in Numbers, is as 959 to 396, or as 17 to 7, nearly.

P R O B. XXXIV.

475. To find when that Part of the Equation of Time, arising from the Obliquity of the Ecliptic to the Equinoctial, is a Maximum.

In the right angled spherical Triangle ABC let the Angle A be that made by the Ecliptic AC, and the Equinoctial AB; then the Problem will be, to find when

when the Difference between the Base AB and the Hypothenufe AC is the greatest possible (the Angle A remaining invariable.) Now (by Art. 254.) we have $\text{Co-f. BC} : \text{Sin. C} :: \text{Fluxion of AC}$



Fluxion of AB : Also (per Spherics) $\text{Sin. C} : \text{Co-f. A} ::$

$\text{Rad.} : \text{Co-f. BC} = \frac{\text{Co-f. A} \times \text{Rad.}}{\text{Sin. C}}$: Whence, by multiplying the two first Terms of the former Proportion

by these equal Quantities, respectively, we get this new Proportion, viz. $\overline{\text{Co-f. BC}}^2 : \text{Co-f. A} \times \text{Radius} ::$ so is the Fluxion of AC to That of AB. But, when AC—AB is a *Maximum*, these Fluxions become equal; and consequently $\overline{\text{Co-f. BC}}^2 = \text{Co-f. A} \times \text{Rad.}$ From which Equation BC, and from thence AC, will be known. Q. E. I.

The same, without Fluxions.

476. It will be (per Spherics) $\text{Rad.} : \text{Co-f. A} :: \text{Tang. AC} : \text{Tang. AB}$; and therefore by Composition and Division, $\text{Rad.} + \text{Co-f. A} : \text{Rad.} - \text{Co-f. A} :: \text{Tang. AC} + \text{Tang. AB} : \text{Tang. AC} - \text{Tang. AB} :: \text{Sin. AC} + \text{AB} : \text{Sin. AC} - \text{AB}$, by the Theorem mention'd in Problem 8th: From which, by following the Steps there laid down, it appears that, $\text{Radius} + \text{Co-f. A} : \text{Radius} - \text{Co-f. A} :: \text{Radius} : \text{Sine of AC} - \text{AB}$, when a *Maximum*: Whence (AC+AB being then = 90°) both AC and BC will be given.

COROLLARY.

477. Since, $\text{Radius} + \text{Co-f. A} : \text{Radius} - \text{Co-f. A} :: \text{Co-tang. } \frac{1}{2} \text{ A} : \text{Tang. } \frac{1}{2} \text{ A}^* :: \overline{\text{Radius}}^2 : \overline{\text{Tang. } \frac{1}{2} \text{ A}}^2$;

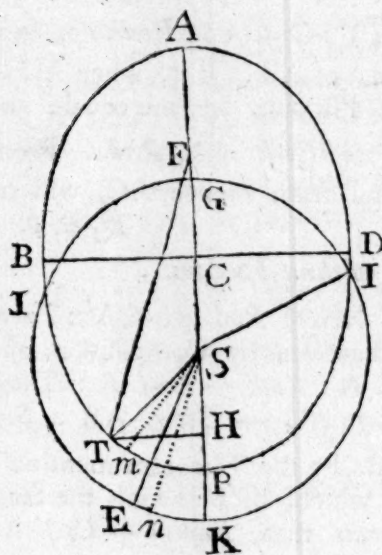
* Vid. p. 70. and 71. of my Trigonometry.

The Resolution of Problems

therefore $\overline{\text{Radius}}^2 : \overline{\text{Tang. } \frac{1}{2} A}^2 :: \text{Radius} : \text{Sine of } \overline{\text{AC} - \text{AB}}$, Or, $\text{Radius} : \text{Tang. } \frac{1}{2} A :: \text{Tang. } \frac{1}{2} A : \text{the Sine of the greatest Equation}$: Which, supposing the Angle A to be $23^\circ 29'$, comes out $2^\circ : 28' : 34''$: answering, in Time, to 9 Minutes : 54 Seconds.

P R O B. XXXV.

478. To determine when the absolute Equation of Time, arising from the Inequality of the Sun's apparent Motion, and the Obliquity of the Ecliptic, conjunctly, is a Maximum.



Lét ABPD be the Ellipsis in which the Earth revolves about the Sun, in the Focus S; let F be the other Focus, and T the Place of the Earth in its Orbit at the Time required. Moreover, about S, as a Center, let a Circle GEKI be described, whose Diameter GK is a Mean Proportional between the two Axes AP and BD of the Ellipsis; so that the Area thereof may be equal to That

of the Ellipsis: And, supposing Sm to be indefinitely near to ST, let ESn be a Sector of the said Circle, equal to the Area TSm.

Then, the Time in which the Earth moves thro' the Arch Tm being to the Time of one intire Revolution, as the Area TSm, or ESn, is to the whole Ellipsis, or the equal Circle GEKF; and these Areas ESn, and GEKI being in the Ratio of the Arch En to the whole Periphery GEKI; it is evident that En,

or

or the Angle ESn , will expreſs the Increate of the *Mean Longitude*, in the foreſaid Time of deſcribing the Arch Tm : And that this Angle or Increate, by reaſon of the Equality of the Areas ESn and TSm , will be to the Angle TSm , expreſſing the correſponding Increate of the *True Longitude*, as ST^2 to SE^2 . Therefore, if the former

be denoted by M , the latter will be represented by $\frac{SE^2}{ST^2}$

$\times M$. But now to get a proper Expreſſion for the Value of this Increate of the *True Longitude*, in Algebraic Terms; let FT be drawn, and alſo TH , perpendicular to AP : Putting $AC (=CP) = a$, $CB = b$, $CS (=CF) = c$, $ST = z$, and the Co-ſine of (TSP) the Earth's Diſtance from its *Perihelion* (to the Radius 1) $= x$: Then FT being $(=AP - ST) = 2a - z$ (by the Property of the Ellipſis) and $SH = xz$ (by *Trig.*)

we have $\overline{FT + ST} \times \overline{FT - ST} (2a \times 2a - 2z) = FS \times 2CH (2c \times 2 \times \overline{c + xz})$ by a known Property of Triangles: From which Equation z (ST) is found $= \frac{a^2 - c^2}{a + cx} = \frac{b^2}{a + cx}$: And this Value, with that of ES^2

$(=ab)$ being ſubſtituted in the Increate of the *True Longitude*, found above, we thence get $\frac{a \times \overline{a + cx}^2}{b^3} \times M$

for the Meaſure of that Increate; where M denotes the Increment of the *Mean Motion* correſponding.

This being obtained, let $\sphericalangle \psi \gamma$ (in the annexed Figure) repreſent the Southern Semi-Circle of the Ecliptic, P the Place of the Perihelion, ψ the Tropic of Capricorn, $\odot$ the apparent Place of the Sun in the Ecliptic, and $Q\odot$ his Declination, at the Time required: Then it appears, (from *Art.* 475.) that the Increate of the *True Longitude* $\sphericalangle \odot$, in an indefinitely ſmall Particle of Time, will be to That of the *Right-Aſcenſion* $\sphericalangle Q$, in the ſame Time, as the Square of the Coſine of $Q\odot$ is to a Rectangle under the Radius and the Co-ſine of the Angle $\sphericalangle$: Therefore, the former,

being expressed by $\frac{a \times \overline{a+cx}^2}{b^3} \times M$, the latter is truly

represented by $\frac{a \times \overline{a+cx}^2}{b^3} \times M \times \frac{\text{Rad.} \times \text{Co-f. } \triangle}{\text{Co-f. } \odot Q^2} :$

Which, in the required Circumstance, when the proposed Equation (or the Difference between the Sun's



Mean Motion and Right Ascension) is a *Maximum*, must consequently be equal to (*M*) the corresponding Increase of *Mean Motion*; and therefore $\frac{a \times \overline{a+cx}^2}{b^3}$

$$= \frac{\overline{\text{Co-f. } \odot Q}^2}{\text{Rad.} \times \text{Co-f. } \triangle}.$$

But, to obtain the Value of the latter Part of this Equation, also, in Algebraic Terms, let the Sine and Co-sine of ($\angle P$) the Distance of the *Perihelion* from $\angle$, be denoted by m and n respectively; then, the Co-sine of $P \odot$ being (as above) expressed by x , and its Sine by $\sqrt{1-xx}$, we shall thence get $nx + m\sqrt{1-xx} = \text{Co-sine of } \odot \angle = \text{Sine of } \triangle \odot$ (by the *Elem. of Trig.*) But (putting the Sine of the Angle $\triangle = p$ and its Co-sine $= q$) we have (per *Spherics*) Radius (1) : Sine $\triangle \odot$ ($nx + m\sqrt{1-xx}$) :: $p : pnx + pm\sqrt{1-xx} = \text{Sine of } Q \odot$; from whence $\overline{\text{Co-f. } \odot Q}^2 = 1 - pnx + pm\sqrt{1-xx}^2$: Which Value, with That of the Co-sine of the Angle $\triangle$, being substituted above, we, at length, get $\frac{a \times \overline{a+cx}^2}{b^3} =$

$$\frac{1 - pnx + pm\sqrt{1 - xx}}{q} ; \text{ from which Equation the}$$

Value of x may be determined.

The foregoing Equation, it may be observed, gives the Time of the *Maximum* which precedes the Winter Solstice; but if the *Maximum* following that Solstice be sought; 'tis but changing the Sign of m , and then you

$$\text{will have } \frac{a \times \overline{a + cx}^2}{b^3} = \frac{1 - pnx - pm\sqrt{1 - xx}}{q},$$

answering in this Case. And from the negative Roots of this, and the preceding, Equation, the Times of the other *Maxima* after, and before, the Summer Solstice will also be obtained.

Q. E. I.

COROLLARY.

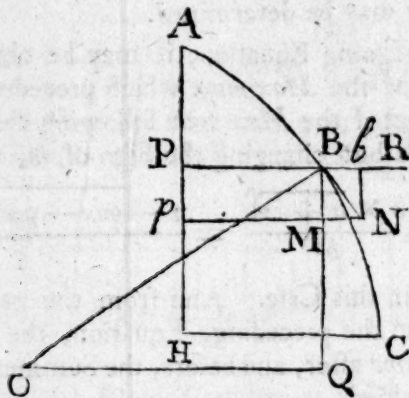
479. It is evident that the Equation of the Earth's Orbit (or that Part of the Equation of Time arising from the Inequality of the Sun's apparent Motion) will be a *Maximum*, when the Center of the Earth is in the Intersection I of the Ellipsis and the Circle; where the *Mean Motion* and *True Longitude* increase with the same Celerity.

P R O B. XXXVI.

480. To determine the Law of the Density of a Medium and the Curve described therein, by Means of an uniform Gravity, so that the Projectile may, every where, move with the same Velocity.

It appears, from Art. 367. that $\sqrt{\frac{yy}{xx}}$ is a general Expression for the Celerity in the Direction of the Ordinate PBR; whence $\frac{z}{y} \times \sqrt{\frac{yy}{xx}}$, or its Equal, $\frac{z}{\sqrt{x}}$, must be the true Measure of the absolute Celerity

lerity, in the Direction BN: Which being a constant Quantity (by Hypothesis) its Square must also be con-



stant, and so, we have $\frac{\dot{x}\dot{z}}{\dot{x}} = a$; and consequently $\dot{x}\dot{x} + \dot{y}\dot{y} (= \dot{z}\dot{z}) = a\dot{x}$.

But, in order to the Solution of the Equation thus given, make $u : 1 :: \dot{x} : \dot{y}$, or $\dot{x} = u\dot{y}$; then $\dot{x} = u\dot{y}$, and, by Substitution, $u^2\dot{y}^2 + \dot{y}^2 = a u\dot{y}$: Hence, $\dot{y}$ being =

$\frac{au}{uu + 1}$, and $\dot{x} = \frac{auu}{uu + 1}$, we get $y = a \times \text{Arch}$, whose

* Art. 142. Tangent is u * (and Secant $\sqrt{1+uu}$); and $x = \frac{1}{2}a \times$

† Art. 126. Hyp. Log. $\sqrt{1+uu} = a \times \text{Hyp. Log. } \sqrt{1+uu}$ †.

Therefore, as the Hyp. Log. of $\sqrt{1+uu}$ is $= \frac{x}{a}$,

the Common Logarithm of $\sqrt{1+uu}$ will be =

$\frac{0.4342944 \text{ } \mathcal{L}c. \times x}{a}$; and consequently $y = a \times \text{Arch}$, whose

Radius is Unity, and Log. Secant $\frac{0.4342944 \text{ } \mathcal{L}c. \times x}{a}$.

Moreover, with respect to the Density of the Medium; if the absolute Force of Gravity, in the Direction QB, be

be denoted by Unity, its Efficacy in the Direction BN, whereby the Body is accelerated, will be expressed by

$\frac{x}{z}$, or its Equal $\frac{u}{\sqrt{1+uu}}$: Which, as the Velocity

is supposed to remain every where the same, must also express the Force of the Resistance, in the opposite Direction, or the true Measure of the required Density. This, therefore, if M be put for the absolute Number whose Hyperbolic Logarithm is Unity, may be had in

Terms of x , and will be $1 - \sqrt[1/2]{\frac{-2x}{M^a}}$: Because

Hyp. Log. $M^{\frac{x}{a}}$ ($= \frac{x}{a}$) being = Hyp. Log. $\sqrt{1+uu}$,

we have $\sqrt{1+uu} = M^{\frac{x}{a}}$; whence $u = M^{\frac{2x}{a}} - 1$, and

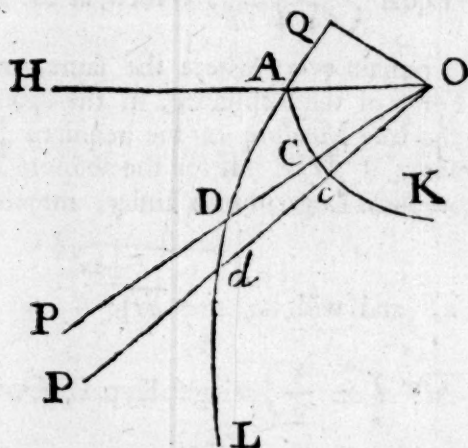
consequently $\frac{u}{\sqrt{1+uu}} = 1 - \sqrt[1/2]{\frac{-2x}{M^a}}$. Q. E. I.

PROB. XXXVII.

481. Let a Line, or an inflexible Rod OP (consider'd without regard to Thickness) be supposed to revolve about one of its Extremes O, as a Center, with a Motion regulated according to any given Law; whilst a Ring, or Ball, carry'd about with it, and tending to the Center O with any given Force, is suffer'd to move or slide, freely along the said Line or Rod: 'Tis propos'd to determine the Velocity of the Ring, and its Pressure upon the Rod, in any propos'd Position, together with the Nature of the Curve ADL described by means of that compound Motion.

Let ODP be any Position of the revolving Line, and D the corresponding Position of the Body: Moreover, supposing ACK to be the Circumference of a Circle

Circle described from the Center O, thro' the given Point A, let the Measure of the angular Celerity of that Line, in the said Circumference ACK, be repre-



sented by u ; also let v denote the Celerity of the Ring at D in the Direction DP; and w the true Measure of the centripetal Force: Call OA, a ; OD, x ; and AC, z ; and let the given Values of u and v , at A, be denoted by b and c respectively. Then it will be, as a :

$x :: u : \left(\frac{ux}{a}\right)$ the paracentric Velocity of the Body, at D; whose Square, divided by the Distance OD, gives

† Art. 211. $\frac{u^2 x}{a^2}$, for the true Measure of the Centrifugal Force † arising from the Revolution of the Rod: From which the centripetal Force w being deducted, the Remainder, $\frac{xu^2}{a^2} - w$, is the true Force whereby the Velocity in the Line OP is accelerated. Therefore (by Art. 218.) we have $v\dot{v} = \frac{xu^2}{a^2} - w \times \dot{x} = \frac{u^2 x \dot{x}}{a^2} - w\dot{x}$.

Moreover, because the Fluxion of the Time is expressed either by $\frac{\dot{x}}{v}$ or by $\frac{\dot{z}}{u}$, these two Values must, there-

therefore, be equal to each other, and consequently

$v = \frac{ux}{z}$: From which, and the preceding Equation

(when u and w are exhibited in Terms of x or z) the quired Relation of v , x and z will also become known— But now, in order to determine the Action of the Rod upon the Ring; let OdP be indefinitely near to ODP , intersecting ADL and ACK in d and c ; and put $Od =$

$x + x'$. Then, because a Body, acted on by no other Force besides That tending to the Center, about which it revolves, describes Areas proportional to the Times*, • Art. 224. and the angular Celerity of a Ray revolving with the Body, is, in that Case, as the Square of the Distance of the Body from the Center, inversely (*vid. Art. 478.*) it follows, that, if the Rod was to cease to act upon the Ring, at the Position ODP , the angular Celerity at c ,

would then be $\frac{x^2}{x + x'} \times u$, instead of $u + u'$. There-

fore the Excess of $u + u'$ above $\frac{x^2}{x + x'} \times u$, which is

$$= u + \frac{2ux'}{x} - \frac{3ux'^2}{x^2} \text{ \&c. is the Increase of the said an-}$$

gular Celerity, at the Distance OC , arising from the Action of the Rod. Therefore it will be, as $OC (a) : OD$

$$(x) :: \text{the said Increase to } \left(\frac{xu'}{a} + \frac{2ux'}{a} - \frac{3ux'^2}{ax} \text{ \&c.} \right)$$

the Alteration of the Ring's paracentric Velocity, arising

from the same Cause. Which, divided by $\left(\frac{x'}{v} \right)$ the

$$\text{Time wherein } It \text{ is produced, gives } \frac{xvu'}{ax} + \frac{2uv}{a} -$$

$\frac{3uvx}{ax}$ &c. for the Measure of the Force, by which *It*

is produced. From whence, by substituting $\frac{u}{x}$ in the

Room of $\frac{u}{x}$, and neglecting all the Terms after the two

• Art. 134. first (in order to have the limiting Ratio *) we get

$\frac{xvu}{ax} + \frac{2uv}{a}$. Therefore it will be, as $\frac{xvu}{ax} + \frac{2uv}{a}$ to

† Art. 211. $\frac{bb}{a}$ †, or as $\frac{xvu}{bbx} + \frac{2uv}{bb}$ to Unity, so is the Action of the Rod upon the Ring, to the (given) Centrifugal Force at A (or the Force that would retain a Body in the Circle ACK, with the Velocity *b*.) *Q. E. I.*

COROLLARY I.

482. If the angular Motion be uniform, the Equations found above, will become $v\dot{v} = \frac{b^2 x \ddot{x}}{a^2} - w\dot{x}$, and $v = \frac{b\dot{x}}{z}$. From the latter of which, by taking the Fluxion, we have $\dot{v} = \frac{b\ddot{x}}{z}$; whence (by Substitution) $\frac{b^2 \dot{x} \ddot{x}}{z\dot{z}} = \frac{b^2 x \ddot{x}}{aa} - w\dot{x}$, and consequently $\ddot{x} - \frac{x\dot{z}^2}{a^2} = -\frac{w\dot{z}^2}{b^2}$; from the Solution of which, the Relation of *x* and *z* will be given. And then, the Value of $v \left(\frac{b\dot{x}}{z} \right)$ being also known, the Action upon the Rod, which in this Case is barely $= \frac{2bv}{a} \left(= \frac{2b^2 \dot{x}}{az} \right)$ will be given likewise,

being

being to $\left(\frac{bb}{a}\right)$ the centrifugal Force in the Circle

ACK, as $\frac{2\dot{x}}{x}$ to Unity.

COROLLARY II.

483. But if the Angular Celerity be proportional to any Power (x^m) of the Distance, and the Centripetal Force w be, also, supposed to vary according to some Power (x^n) of the same Distance: Then, putting p to denote the Centripetal, and q the Centrifugal, Force, at the given Point A, the Value of w will, here, be ex-

pounded by $\frac{x^n}{a^n} \times p$, and That of u by $\frac{x^m}{a^m} \times b$: And there-

fore, the paracentric Velocity of the Ring at D being =

$$\frac{x^m}{a^m} \times b \times \frac{x}{a} \left(= \frac{bx^{m+1}}{a^{m+1}} \right) \text{ it will be as } \frac{bb}{a} : \frac{b^2 x^{2m+2}}{xa^{2m+2}}$$

$$\therefore q : \frac{x^{2m+1}}{a^{2m+1}} \times q, \text{ the Centrifugal Force at D}^*. \text{ Hence }^* \text{ Art. 211,}$$

$$v\dot{v} = \frac{qx^{2m+1}\dot{x}}{a^{2m+1}} - \frac{px^{n+1}\dot{x}}{a^n}; \text{ whereof the (corrected) Fluent}$$

$$\text{is } \frac{1}{2} v\dot{v} - \frac{1}{2} c\dot{c} = \frac{qx^{2m+2}}{2m+2 \times a^{2m+1}} - \frac{px^{n+1}}{n+1 \times a^n} - \frac{qa}{2m+1}$$

$$+ \frac{pa}{n+1}: \text{ From whence } v \text{ is found } =$$

$$\sqrt{cc - \frac{qa}{m+1} + \frac{2pa}{n+1} + \frac{qx^{2m+2}}{m+1 \cdot a^{2m+1}} - \frac{2px^{n+1}}{n+1 \cdot a^n}}$$

$$\text{and } z \left(= \frac{u\dot{x}}{v} = \frac{bx^{m+1}\dot{x}}{a^{m+1}v} \right) =$$

$$\frac{bx^{m+1}\dot{x}}{a^{m+1} \sqrt{cc - \frac{qa}{m+1} + \frac{2pa}{n+1} + \frac{qx^{2m+2}}{m+1 \cdot a^{2m+1}} - \frac{2px^{n+1}}{n+1 \cdot a^n}}}$$

O o

More.

Moreover, by substituting for u , and its Fluxion, we get $\frac{xvu}{ax} + \frac{2uv}{a} = \frac{m+2}{m+1} \times \frac{bx^mv}{a^{m+1}}$, expressing the Action

of the Rod upon the Ring: Which, therefore, when m is expounded by -2 , will intirely vanish: And, in that Case, $\dot{z}$ will become =

$$\frac{a^2 b \dot{x}}{x \sqrt{cc + qa + \frac{2pa}{n+1} \times x^2 - qa^3 - \frac{2px^{n+3}}{n+1 \cdot a^n}}}$$

expressing the Nature of the Trajectory described by means of a Centripetal Force, varying according to any Power (x^n) of the Distance. But this Equation will be render'd somewhat more commodious, by substituting the Values of b and c : For, if OQ (perpendicular to the Tangent at A) be denoted by b , it will be, $b : \sqrt{a^2 - b^2}$ (AQ) :: b (the Celerity in the Direction AC)

* Art. 35. to $c = \frac{b \sqrt{a^2 - b^2}}{b} =$ the Celerity in the Direction AH *.

† Art. 211. Therefore, b being $= \sqrt{aq}$ †, we have $c^2 = \frac{a^3 q}{bb} - aq$;

$$\text{and } \dot{z} = \frac{a^2 \dot{x}}{x \sqrt{\frac{aa}{bb} + \frac{2p}{n+1 \cdot q} \times x^2 - a^2 - \frac{2px^{n+3}}{n+1 \cdot qa^{n+1}}}}$$

Which Equation is the same, in effect, with that given in Art. 242. by a different Method.

COROLLARY III.

484. If the Angular Celerity be supposed uniform, and the Ring to have no other Motion along the Rod than what it acquires from its Centrifugal Force; then c , m and p being all of them equal to Nothing, $\dot{z}$ will here become, barely =

$$\frac{b \dot{x}}{\sqrt{-qa + \frac{qx^2}{a}}} = \frac{a \dot{x}}{\sqrt{x^2 - a^2}} : \text{And}$$

there-

therefore $z = a \times \text{Hyp. Log.} \frac{x + \sqrt{xx - aa}}{a}$. Hence

if the Number whose Hyp. Log. is $\frac{z}{a}$ be denoted by

N , we shall have $\frac{x + \sqrt{xx - aa}}{a} = N$: From which

x is found $= a \times \frac{N}{2} + \frac{1}{2N}$; whence $\dot{x}$ is, also, had $=$

$\frac{a\dot{N}}{2} - \frac{a\dot{N}}{2N^2} = \frac{N\dot{z}}{2} - \frac{\dot{z}}{2N}$ (because $\frac{\dot{N}}{N} = \frac{\dot{z}}{a}$). There-

fore, it will be (by *Corol. 1.*) as Unity is to $\frac{N}{2} - \frac{1}{2N}$,

so is the Angular Velocity (b) in the Arch ACK to the Velocity with which the Body recedes from the Center of Motion: And so, likewise, is the Centrifugal Force in that Arch to half the Pressure upon the Rod—By

taking $z =$ the whole Periphery, or $\frac{z}{a} = 2 \times 3.145$

&c. N will come out $= 535.5$, and $x = 267.7 \times a$: From whence it appears that the Distance of the Ring from the Center at the End of one intire Revolution will be almost 268 times as great as at first.

COROLLARY IV.

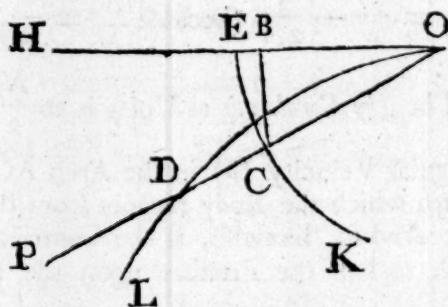
485. If a Body be supposed to descend from the Point O, (*see the next Fig.*) by the Force of its own Gravity, along an inclin'd Plane OCP; whilst the Plane itself moves uniformly about that Point, from an horizontal Position OEH; then the Place, and the Pressure of the Body upon the Plane, in any given Position OCP, may, also, be derived from the Equations in *Corollary 1.* For let CB (perpendicular to OH) be put $= y$; and let the Ratio of the Centrifugal Force in the Circle ECK, to the Force of Gravity (*given by Art. 217.*) be as r to Unity: Then, as

the Measure of the former Force is expressed by $\frac{bb}{a}$,

O o z

That

That of the latter must be represented by $\frac{bb}{ra}$; and, consequently, its Efficacy in the Direction PO, by $\frac{bby}{raa} \left(= \frac{bb}{ra} \times \frac{CB}{OC} \right)$: Which Value being substituted for $-w$, in the aforefaid Corollary, we have $\ddot{x} - \frac{x\dot{z}^2}{aa} = \frac{y\dot{z}^2}{raa}$. But now, in order to the Solution of this



Equation, put the Radius OC (a) = 1 (that the Operation may be as simple as possible) also, instead of y ,

* Art. 425, let its Equal $z - \frac{z^3}{2 \cdot 3} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5}$ * &c. be substituted, and let x be assumed = $Az^3 + Bz^5 + Cz^7 + Dz^9$ &c.

Then, by proceeding as is taught in Art. 267. the

Value of x will come out = $\frac{1}{r}$ into $\frac{z^3}{2 \cdot 3} + \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}$

+ $\frac{z^{11}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11}$ + &c. Whence

the Velocity $\left(\frac{b\dot{x}}{z} \right)$ in the Plane, is, also, found =

$\frac{b}{r}$ into $\frac{z^2}{2} + \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$ &c. Which, therefore, is

to

to (b) the angular Velocity of the Plane, in the Arch

ECK, as $\frac{z^2}{2} + \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \text{Ec. to } r$. Moreover,

the Centrifugal Force in the said Arch being denoted by r (the Force of Gravity being Unity) it will likewise be

(by the above-mention'd *Corol.*) as $1 : \frac{2\dot{x}}{z} :: r : \left(\frac{2r\dot{x}}{z} = \right)$

$z^2 + \frac{z^6}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{z^{10}}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \text{Ec.} = \text{the}$

Force sufficient to keep the Body upon the Plane. But the Force of Gravity in a Direction perpendicular to the Plane (the Weight of the Body being represented

by Unity) is $\frac{OB}{OC} = 1 - \frac{z^2}{2} + \frac{z^4}{2 \cdot 3 \cdot 4} * \text{Ec.}$ From * Art. 425.

which deducting the Quantity last found, there rests $1 -$

$\frac{3z^2}{2} + \frac{z^4}{2 \cdot 3 \cdot 4} - \frac{3z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \text{ Ec.}$ for the true Pref-

sure of the Body upon the Plane. By putting of Which equal to Nothing, z^2 will be found $= 0.67715$; answering to an Angle (EOC) of $47^\circ : 9'$: Which Angle is therefore the Inclination, when the Force of Gravity is no longer sufficient to keep the Body upon the Plane.

Though the Value of x , given above, is found by an Infinite Series, yet the Sum of that Series is easily exhibited by the Measures of Angles and Ratios. For, putting N to denote the Number whose hyperbolic Logarithm is z ,

$$\text{we have} \left\{ \begin{array}{l} 1 + z + \frac{z^2}{2} + \frac{z^3}{2 \cdot 3} + \frac{z^4}{2 \cdot 3 \cdot 4} \text{ Ec.} = N \\ 1 - z + \frac{z^2}{2} - \frac{z^3}{2 \cdot 3} + \frac{z^4}{2 \cdot 3 \cdot 4} \text{ Ec.} = \frac{1}{N} \end{array} \right.$$

* Art. 424.

Half the Difference of which two Equations is $z +$

$$\frac{z^3}{2 \cdot 3} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} \text{ Ec.} = \frac{N}{2} - \frac{1}{2N} :$$

O o 3

From

From which taking $z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \frac{z^7}{2.3.4.5.6.7}$

&c. = y ; and dividing the Remainder by $2r$, there re-

sults $\left(\frac{1}{r} \times \frac{z^3}{2.3} + \frac{z^7}{2.3.4.5.6.7} \text{ \&c.} \right) \frac{1}{2r} \times$

$\frac{N}{2} - \frac{1}{2N} - y$, for the true Value of x . Which, if

required, may be expressed independent of r ; by putting d for the Distance thro' which a Body, freely, descends in the first Second of Time, and taking b to denote the Velocity of the Plane, (*per Second*) in the Arch ECK: For then, the Ratio of the Centrifugal Force, in the said Arch, to the Force of Gravity (or

• Art. 211. That of r to 1) being as $\frac{bb}{1} (= \frac{bb}{OC})$ to $2d$ *,

we shall have $r = \frac{bb}{2d}$, and consequently $x = \frac{d}{bb} \times$

$\frac{N}{2} - \frac{1}{2N} - y$.

By Computations, not very unlike Those above, the Motion of the Moon's *Apogee*, and the principal Equations of the Lunar Orbit may be exhibited; by means of proper Approximations, derived from the general Equations in Art. 481. But this is a Consideration that would require a Volume of itself, to treat it, from first Principles, with all the Attention and Perspicuity suitable to the Importance of the Subject. I shall conclude this Work with the following short Table of *Hyperbolical Logarithms*, drawn up and communicated by my ingenious Friend Mr. *John Turner*: Whereof the Use, in finding Fluents, will sufficiently appear from the foregoing Pages. In the said Table we have given the Hyperbolical Logarithms of every whole Number and hundredth Part of an Unit, from 1 to 10 (which Form is best adapted to the Purposes above-mention'd) by Help whereof, and the following Observations, the Hyperbolical

perbolical Logarithm of any Number, not exceeding seven Places of Figures, may be found with very little Trouble.

1°. *If the Number given be between 1 and 10 (so as to fall within the Limits of the Table.)*

Then take from it the next inferior Number in the Table, and divide the Remainder by the said inferior Number increased by half the Remainder; and let the Quotient be added to the Logarithm of the said inferior Number, the Sum will be the Logarithm sought.

Thus, let the Hyperbolical Logarithm of 3.45678 be required; then the Operation will stand thus: $3.45339).005678(.0016442$: Which added to 1.2385742, the Log. of 3.45, gives 1.2400184 for the Logarithm sought.

2°. *When the Number proposed exceeds 10.*

Find the Logarithm thereof, supposing all the Figures after the First to be Decimals; then to the Logarithm, so found, let 2.3025851, 4.6051702, or 6.9077553 &c. be added, according as the whole Number consists of 2, 3, or 4 &c. Places: The Sum will be the Logarithm sought.

Thus, the Hyperbolical Logarithm of 345.678 will be found to be 5.8451886: For That of 3.45678 being 1.2400184; the same, added to 4.6051702, gives the very Quantity above exhibited. The Reason of which, as well as of the Operation in the preceding Case, is evident from the Nature and Construction of Logarithms.

N	Logarithm	N	Logarithm	N	Logarithm
1.01	.0099503	1.34	.2926696	1.67	.5128236
1.02	.0198026	1.35	.3001045	1.68	.5187937
1.03	.0295588	1.36	.3074846	1.69	.5247285
1.04	.0392207	1.37	.3148107	1.70	.5306282
1.05	.0487902	1.38	.3220834	1.71	.5364933
1.06	.0582689	1.39	.3293037	1.72	.5423242
1.07	.0676586	1.40	.3364722	1.73	.5481214
1.08	.0769610	1.41	.3435897	1.74	.5538851
1.09	.0861777	1.42	.3506568	1.75	.5596157
1.10	.0953102	1.43	.3576744	1.76	.5653138
1.11	.1043600	1.44	.3646431	1.77	.5709795
1.12	.1133287	1.45	.3715635	1.78	.5766133
1.13	.1222176	1.46	.3784364	1.79	.5822156
1.14	.1310283	1.47	.3852624	1.80	.5877866
1.15	.1397619	1.48	.3920420	1.81	.5933268
1.16	.1484200	1.49	.3987761	1.82	.5988365
1.17	.1570037	1.50	.4054651	1.83	.6043159
1.18	.1655144	1.51	.4121096	1.84	.6097655
1.19	.1739533	1.52	.4187103	1.85	.6151856
1.20	.1823215	1.53	.4252677	1.86	.6205764
1.21	.1906203	1.54	.4317824	1.87	.6259384
1.22	.1988508	1.55	.4382549	1.88	.6312717
1.23	.2070141	1.56	.4446858	1.89	.6365768
1.24	.2151113	1.57	.4510756	1.90	.6418538
1.25	.2231435	1.58	.4574248	1.91	.6471032
1.26	.2311117	1.59	.4637340	1.92	.6523251
1.27	.2390169	1.60	.4700036	1.93	.6575200
1.28	.2468600	1.61	.4762341	1.94	.6626879
1.29	.2546422	1.62	.4824261	1.95	.6678293
1.30	.2623642	1.63	.4885800	1.96	.6729444
1.31	.2700271	1.64	.4946962	1.97	.6780335
1.32	.2776317	1.65	.5007752	1.98	.6830968
1.33	.2851789	1.66	.5068175	1.99	.6881346
1.34	.2926696	1.67	.5128236	2.00	.6931472



N	Logarithm	N	Logarithm	N	Logarithm
2.01	.6981347	2.34	.8501509	2.67	.9820784
2.02	.7030974	2.35	.8544153	2.68	.9858167
2.03	.7080357	2.36	.8586616	2.69	.9895411
2.04	.7129497	2.37	.8628899	2.70	.9932517
2.05	.7178397	2.38	.8671004	2.71	.9969486
2.06	.7227059	2.39	.8712933	2.72	1.0006318
2.07	.7275485	2.40	.8754687	2.73	1.0043015
2.08	.7323678	2.41	.8796267	2.74	1.0079579
2.09	.7371640	2.42	.8837675	2.75	1.0116008
2.10	.7419373	2.43	.8878912	2.76	1.0152306
2.11	.7466879	2.44	.8919980	2.77	1.0188473
2.12	.7514160	2.45	.8960880	2.78	1.0224509
2.13	.7561219	2.46	.9001613	2.79	1.0260415
2.14	.7608058	2.47	.9042181	2.80	1.0296194
2.15	.7654678	2.48	.9082585	2.81	1.0331844
2.16	.7701082	2.49	.9122826	2.82	1.0367368
2.17	.7747271	2.50	.9162907	2.83	1.0402766
2.18	.7793248	2.51	.9202827	2.84	1.0438040
2.19	.7839015	2.52	.9242589	2.85	1.0473189
2.20	.7884573	2.53	.9282193	2.86	1.0508216
2.21	.7929925	2.54	.9321640	2.87	1.0543120
2.22	.7975071	2.55	.9360933	2.88	1.0577902
2.23	.8020015	2.56	.9400072	2.89	1.0612564
2.24	.8064758	2.57	.9439058	2.90	1.0647107
2.25	.8109302	2.58	.9477893	2.91	1.0681530
2.26	.8153648	2.59	.9516578	2.92	1.0715836
2.27	.8197798	2.60	.9555114	2.93	1.0750024
2.28	.8241754	2.61	.9593502	2.94	1.0784095
2.29	.8285518	2.62	.9631743	2.95	1.0818051
2.30	.8329091	2.63	.9669838	2.96	1.0851892
2.31	.8372475	2.64	.9707789	2.97	1.0885619
2.32	.8415671	2.65	.9745596	2.98	1.0919233
2.33	.8458682	2.66	.9783261	2.99	1.0952733
2.34	.8501509	2.67	.9820784	3.00	1.0986123

N	Logarithm	N	Logarithm	N	Logarithm
3.01	1.1019400	3.34	1.2059707	3.67	1.3001916
3.02	1.1052568	3.35	1.2089603	3.68	1.3029127
3.03	1.1085626	3.36	1.2119409	3.69	1.3056264
3.04	1.1118575	3.37	1.2149127	3.70	1.3083328
3.05	1.1151415	3.38	1.2178757	3.71	1.3110318
3.06	1.1184149	3.39	1.2208299	3.72	1.3137236
3.07	1.1216775	3.40	1.2237754	3.73	1.3164082
3.08	1.1249295	3.41	1.2267122	3.74	1.3190856
3.09	1.1281710	3.42	1.2296405	3.75	1.3217558
3.10	1.1314021	3.43	1.2325605	3.76	1.3244189
3.11	1.1346227	3.44	1.2354714	3.77	1.3270749
3.12	1.1378330	3.45	1.2383742	3.78	1.3297240
3.13	1.1410330	3.46	1.2412685	3.79	1.3323660
3.14	1.1442227	3.47	1.2441545	3.80	1.3350010
3.15	1.1474024	3.48	1.2470322	3.81	1.3376291
3.16	1.1505720	3.49	1.2499017	3.82	1.3402504
3.17	1.1537315	3.50	1.2527629	3.83	1.3428648
3.18	1.1568811	3.51	1.2556160	3.84	1.3454723
3.19	1.1600209	3.52	1.2584609	3.85	1.3480731
3.20	1.1631508	3.53	1.2612978	3.86	1.3506671
3.21	1.1662709	3.54	1.2641266	3.87	1.3532544
3.22	1.1693813	3.55	1.2669475	3.88	1.3558351
3.23	1.1724821	3.56	1.2697605	3.89	1.3584091
3.24	1.1755733	3.57	1.2725655	3.90	1.3609765
3.25	1.1786549	3.58	1.2753627	3.91	1.3635373
3.26	1.1817271	3.59	1.2781521	3.92	1.3660916
3.27	1.1847899	3.60	1.2809338	3.93	1.3686394
3.28	1.1878434	3.61	1.2837077	3.94	1.3711807
3.29	1.1908875	3.62	1.2864740	3.95	1.3737156
3.30	1.1939224	3.63	1.2892326	3.96	1.3762440
3.31	1.1969481	3.64	1.2919836	3.97	1.3787661
3.32	1.1999647	3.65	1.2947271	3.98	1.3812818
3.33	1.2029722	3.66	1.2974631	3.99	1.3837912
3.34	1.2059707	3.67	1.3001916	4.00	1.3862943

N	Logarithm	N	Logarithm	N	Logarithm
4.01	1.3887912	4.34	1.4678743	4.67	1.5411590
4.02	1.3912818	4.35	1.4701758	4.68	1.5432981
4.03	1.3937663	4.36	1.4724720	4.69	1.5454325
4.04	1.3962446	4.37	1.4747630	4.70	1.5475625
4.05	1.3987168	4.38	1.4770487	4.71	1.5496879
4.06	1.4011829	4.39	2.4793292	4.72	1.5518087
4.07	1.4036429	4.40	1.4816045	4.73	1.5539252
4.08	1.4060969	4.41	1.4838746	4.74	1.5560371
4.09	1.4085449	4.42	1.4861396	4.75	1.5581446
4.10	1.4109869	4.43	1.4883995	4.76	1.5602476
4.11	1.4134230	4.44	1.4906543	4.77	1.5623462
4.12	1.4158531	4.45	1.4929040	4.78	1.5644405
4.13	1.4182774	4.46	1.4951487	4.79	1.5665304
4.14	1.4206957	4.47	1.4973883	4.80	1.5686159
4.15	1.4231083	4.48	1.4996230	4.81	1.5706971
4.16	1.4255150	4.49	1.5018527	4.82	1.5727739
4.17	1.4279160	4.50	1.5040774	4.83	1.5748464
4.18	1.4303112	4.51	1.5062971	4.84	1.5769147
4.19	1.4327007	4.52	1.5085119	4.85	1.5789787
4.20	1.4350845	4.53	1.5107219	4.86	1.5810384
4.21	1.4374626	4.54	1.5129269	4.87	1.5830939
4.22	1.4398351	4.55	1.5151272	4.88	1.5851452
4.23	1.4422020	4.56	1.5173226	4.89	1.5871923
4.24	1.4445632	4.57	1.5195132	4.90	1.5892352
4.25	1.4469189	4.58	1.5216990	4.91	1.5912739
4.26	1.4492691	4.59	1.5238800	4.92	1.5933085
4.27	1.4516138	4.60	1.5260563	4.93	1.5953389
4.28	1.4539530	4.61	1.5282278	4.94	1.5973653
4.29	1.4562867	4.62	1.5303947	4.95	1.5993875
4.30	1.4586149	4.63	1.5325568	4.96	1.6014057
4.31	1.4609379	4.64	1.5347143	4.97	1.6034198
4.32	1.4632553	4.65	1.5368672	4.98	1.6054298
4.33	1.4655675	4.66	1.5390154	4.99	1.6074358
4.34	1.4678743	4.67	1.5411590	5.00	1.6094379

N	Logarithm	N	Logarithm	N	Logarithm
5.01	1.6114359	5.34	1.6752256	5.67	1.7351891
5.02	1.6134300	5.35	1.6770965	5.68	1.7369512
5.03	1.6154200	5.36	1.6789639	5.69	1.7387102
5.04	1.6174060	5.37	1.6808278	5.70	1.7404661
5.05	1.6193882	5.38	1.6826882	5.71	1.7422189
5.06	1.6213664	5.39	1.6845453	5.72	1.7439687
5.07	1.6233408	5.40	1.6863989	5.73	1.7457155
5.08	1.6253112	5.41	1.6882491	5.74	1.7474591
5.09	1.6272778	5.42	1.6900958	5.75	1.7491998
5.10	1.6292405	5.43	1.6919391	5.76	1.7509374
5.11	1.6311994	5.44	1.6937790	5.77	1.7526720
5.12	1.6331544	5.45	1.6956155	5.78	1.7544036
5.13	1.6351056	5.46	1.6974487	5.79	1.7561323
5.14	1.6370530	5.47	1.6992786	5.80	1.7578579
5.15	1.6389967	5.48	1.7011051	5.81	1.7595805
5.16	1.6409365	5.49	1.7029282	5.82	1.7613002
5.17	1.6428726	5.50	1.7047481	5.83	1.7630170
5.18	1.6448050	5.51	1.7065646	5.84	1.7647308
5.19	1.6467336	5.52	1.7083778	5.85	1.7664416
5.20	1.6486586	5.53	1.7101878	5.86	1.7681496
5.21	1.6505798	5.54	1.7119944	5.87	1.7698546
5.22	1.6524974	5.55	1.7137979	5.88	1.7715567
5.23	1.6544112	5.56	1.7155981	5.89	1.7732559
5.24	1.6563214	5.57	1.7173950	5.90	1.7749523
5.25	1.6582280	5.58	1.7191887	5.91	1.7766458
5.26	1.6601310	5.59	1.7209792	5.92	1.7783364
5.27	1.6620303	5.60	1.7227666	5.93	1.7800242
5.28	1.6639260	5.61	1.7245507	5.94	1.7817091
5.29	1.6658182	5.62	1.7263316	5.95	1.7833912
5.30	1.6677068	5.63	1.7281094	5.96	1.7850704
5.31	1.6695918	5.64	1.7298840	5.97	1.7867469
5.32	1.6714733	5.65	1.7316555	5.98	1.7884205
5.33	1.6733512	5.66	1.7334238	5.99	1.7900914
5.34	1.6752256	5.67	1.7351891	6.00	1.7917594

N	Logarithm	N	Logarithm	N	Logarithm
6.01	1.7934247	6.34	1.8468787	6.67	1.8976198
6.02	1.7950872	6.35	1.8484547	6.68	1.8991179
6.03	1.7967470	6.36	1.8500283	6.69	1.9006138
6.04	1.7984040	6.37	1.8515994	6.70	1.9021075
6.05	1.8000582	6.38	1.8531680	6.71	1.9035989
6.06	1.8017098	6.39	1.8547342	6.72	1.9050881
6.07	1.8033586	6.40	1.8562979	6.73	1.9065751
6.08	1.8050047	6.41	1.8578592	6.74	1.9080600
6.09	1.8066481	6.42	1.8594181	6.75	1.9095425
6.10	1.8082887	6.43	1.8609745	6.76	1.9110228
6.11	1.8099267	6.44	1.8625285	6.77	1.9125011
6.12	1.8115621	6.45	1.8640801	6.78	1.9139771
6.13	1.8131947	6.46	1.8656293	6.79	1.9154509
6.14	1.8148247	6.47	1.8671761	6.80	1.9169226
6.15	1.8164520	6.48	1.8687205	6.81	1.9183921
6.16	1.8180767	6.49	1.8702625	6.82	1.9198594
6.17	1.8196988	6.50	1.8718021	6.83	1.9213247
6.18	1.8213182	6.51	1.8733394	6.84	1.9227877
6.19	1.8229351	6.52	1.8748743	6.85	1.9242486
6.20	1.8245493	6.53	1.8764069	6.86	1.9257074
6.21	1.8261608	6.54	1.8779371	6.87	1.9271641
6.22	1.8277699	6.55	1.8794650	6.88	1.9286186
6.23	1.8293763	6.56	1.8809906	6.89	1.9300710
6.24	1.8309801	6.57	1.8825138	6.90	1.9315214
6.25	1.8325814	6.58	1.8840347	6.91	1.9329696
6.26	1.8341801	6.59	1.8855533	6.92	1.9344157
6.27	1.8357763	6.60	1.8870696	6.93	1.9358598
6.28	1.8373699	6.61	1.8885837	6.94	1.9373017
6.29	1.8389610	6.62	1.8900954	6.95	1.9387416
6.30	1.8405496	6.63	1.8916048	6.96	1.9401794
6.31	1.8421356	6.64	1.8931119	6.97	1.9416152
6.32	1.8437191	6.65	1.8946168	6.98	1.9430489
6.33	1.8453002	6.66	1.8961194	6.99	1.9444805
6.34	1.8468787	6.67	1.8976198	7.00	1.9459101

N	Logarithm	N	Logarithm	N	Logarithm
7.01	1.9473376	7.34	1.9933387	7.67	2.0373166
7.02	1.9487632	7.35	1.9947002	7.68	2.0386195
7.03	1.9501866	7.36	1.9960599	7.69	2.0399207
7.04	1.9516080	7.37	1.9974177	7.70	2.0412203
7.05	1.9530275	7.38	1.9987736	7.71	2.0425181
7.06	1.9544449	7.39	2.0001278	7.72	2.0438143
7.07	1.9558604	7.40	2.0014800	7.73	2.0451088
7.08	1.9572739	7.41	2.0028305	7.74	2.0464016
7.09	1.9586853	7.42	2.0041790	7.75	2.0476928
7.10	1.9600947	7.43	2.0055258	7.76	2.0489823
7.11	1.9615022	7.44	2.0068708	7.77	2.0502701
7.12	1.9629077	7.45	2.0082140	7.78	2.0515563
7.13	1.9643112	7.46	2.0095553	7.79	2.0528408
7.14	1.9657127	7.47	2.0108949	7.80	2.0541237
7.15	1.9671123	7.48	2.0122327	7.81	2.0554049
7.16	1.9685099	7.49	2.0135687	7.82	2.0566845
7.17	1.9699056	7.50	2.0149030	7.83	2.0579624
7.18	1.9712993	7.51	2.0162354	7.84	2.0592388
7.19	1.9726911	7.52	2.0175661	7.85	2.0605135
7.20	1.9740810	7.53	2.0188950	7.86	2.0617866
7.21	1.9754689	7.54	2.0202221	7.87	2.0630580
7.22	1.9768549	7.55	2.0215475	7.88	2.0643278
7.23	1.9782390	7.56	2.0228711	7.89	2.0655961
7.24	1.9796212	7.57	2.0241929	7.90	2.0668627
7.25	1.9810014	7.58	2.0255131	7.91	2.0681277
7.26	1.9823798	7.59	2.0268315	7.92	2.0693911
7.27	1.9837562	7.60	2.0281482	7.93	2.0706530
7.28	1.9851308	7.61	2.0294631	7.94	2.0719132
7.29	1.9865035	7.62	2.0307763	7.95	2.0731719
7.30	1.9878743	7.63	2.0320878	7.96	2.0744290
7.31	1.9892432	7.64	2.0333976	7.97	2.0756845
7.32	1.9906103	7.65	2.0347056	7.98	2.0769384
7.33	1.9919754	7.66	2.0360119	7.99	2.0781907
7.34	1.9933387	7.67	2.0373166	8.00	2.0794415

N	Logarithm	N	Logarithm	N	Logarithm
8.01	2.0806907	8.34	2.1210632	8.67	2.1598687
8.02	2.0819384	8.35	2.1222615	8.68	2.1610215
8.03	2.0831845	8.36	2.1234584	8.69	2.1621729
8.04	2.0844290	8.37	2.1246539	8.70	2.1633230
8.05	2.0856720	8.38	2.1258479	8.71	2.1644718
8.06	2.0869135	8.39	2.1270405	8.72	2.1656192
8.07	2.0881534	8.40	2.1282317	8.73	2.1667653
8.08	2.0893918	8.41	2.1294214	8.74	2.1679101
8.09	2.0906287	8.42	2.1306098	8.75	2.1690536
8.10	2.0918640	8.43	2.1317967	8.76	2.1701959
8.11	2.0930984	8.44	2.1329822	8.77	2.1713367
8.12	2.0943306	8.45	2.1341664	8.78	2.1724763
8.13	2.0955613	8.46	2.1353491	8.79	2.1736146
8.14	2.0967905	8.47	2.1365304	8.80	2.1747517
8.15	2.0980182	8.48	2.1377104	8.81	2.1758874
8.16	2.0992444	8.49	2.1388889	8.82	2.1770218
8.17	2.1004691	8.50	2.1400661	8.83	2.1781550
8.18	2.1016923	8.51	2.1412419	8.84	2.1792868
8.19	2.1029140	8.52	2.1424163	8.85	2.1804174
8.20	2.1041341	8.53	2.1435893	8.86	2.1815467
8.21	2.1053529	8.54	2.1447609	8.87	2.1826747
8.22	2.1065702	8.55	2.1459312	8.88	2.1838015
8.23	2.1077861	8.56	2.1471001	8.89	2.1849270
8.24	2.1089998	8.57	2.1482676	8.90	2.1860512
8.25	2.1102128	8.58	2.1494339	8.91	2.1871742
8.26	2.1114243	8.59	2.1505987	8.92	2.1882959
8.27	2.1126343	8.60	2.1517622	8.93	2.1894163
8.28	2.1138428	8.61	2.1529243	8.94	2.1905355
8.29	2.1150499	8.62	2.1540851	8.95	2.1916535
8.30	2.1162555	8.63	2.1552445	8.96	2.1927702
8.31	2.1174596	8.64	2.1564026	8.97	2.1938856
8.32	2.1186622	8.65	2.1575593	8.98	2.1949998
8.33	2.1198634	8.66	2.1587147	8.99	2.1961128
8.34	2.1210632	8.67	2.1598687	9.00	2.1972245

N	Logarithm	N	Logarithm	N	Logarithm
9.01	2.1983350	9.34	2.2343062	9.67	2.2690282
9.02	2.1994443	9.35	2.2353763	9.68	2.2700618
9.03	2.2005523	9.36	2.2364452	9.69	2.2710944
9.04	2.2016591	9.37	2.2375130	9.70	2.2721258
9.05	2.2027647	9.38	2.2385797	9.71	2.2731562
9.06	2.2038691	9.39	2.2396452	9.72	2.2741856
9.07	2.2049722	9.40	2.2407096	9.73	2.2752138
9.08	2.2060741	9.41	2.2417729	9.74	2.2762411
9.09	2.2071748	9.42	2.2428350	9.75	2.2772673
9.10	2.2082744	9.43	2.2438960	9.76	2.2782924
9.11	2.2093727	9.44	2.2449559	9.77	2.2793165
9.12	2.2104697	9.45	2.2460147	9.78	2.2803395
9.13	2.2115656	9.46	2.2470723	9.79	2.2813614
9.14	2.2126603	9.47	2.2481288	9.80	2.2823823
9.15	2.2137538	9.48	2.2491843	9.81	2.2834022
9.16	2.2148461	9.49	2.2502386	9.82	2.2844211
9.17	2.2159372	9.50	2.2512917	9.83	2.2854389
9.18	2.2170272	9.51	2.2523438	9.84	2.2864556
9.19	2.2181160	9.52	2.2533948	9.85	2.2874714
9.20	2.2192034	9.53	2.2544446	9.86	2.2884861
9.21	2.2202898	9.54	2.2554934	9.87	2.2894998
9.22	2.2213750	9.55	2.2565411	9.88	2.2905124
9.23	2.2224590	9.56	2.2575877	9.89	2.2915241
9.24	2.2235418	9.57	2.2586332	9.90	2.2925347
9.25	2.2246235	9.58	2.2596776	9.91	2.2935443
9.26	2.2257040	9.59	2.2607209	9.92	2.2945529
9.27	2.2267833	9.60	2.2617631	9.93	2.2955604
9.28	2.2278615	9.61	2.2628042	9.94	2.2965670
9.29	2.2289385	9.62	2.2638442	9.95	2.2975725
9.30	2.2300144	9.63	2.2648832	9.96	2.2985770
9.31	2.2310890	9.64	2.2659211	9.97	2.2995806
9.32	2.2321626	9.65	2.2669579	9.98	2.3005831
9.33	2.2332350	9.66	2.2679936	9.99	2.3015846
9.34	2.2343062	9.67	2.2690282	10.00	2.3025851

F I N I S.

